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HISTORY

*Loria, Gino. **Storia delle matematiche dall'alba della civiltà al secolo XIX.** 2d ed. Ulrico Hoepli, Milano, 1950. xxxv+975 pp. 3800 Lire.

This is a reprint of the author's well-known work [Società Typografico-Editrice Nazionale, Torino, v. 1, 1929, v. 2, 1931, v. 3, 1933]. In a few cases some attention has been paid to publications having appeared since the first edition. Generally speaking, however, no changes were made, even most of the printer's errors being reproduced faithfully. The scarce additions to the lists of references bristle with orthographical mistakes. *E. J. Dijksterhuis.*

*Michel, Paul-Henri. **De Pythagore à Euclide. Contribution à l'histoire des mathématiques préeuclidiennes.** Société d'Édition "Les Belles Lettres," Paris, 1950. 699 pp.

This work is a compilation of the literature on early Greek mathematics brought up to date until 1946 as stated on p. 7. Factually it represents the state of the problem around 1900 when neither the oriental field nor astronomy had seriously entered the picture. The absence of an index makes it difficult to utilize the bibliographical material.

O. Neugebauer (Providence, R. I.).

Baqir, Taha. An important mathematical problem text from Tell Harmal. (On a Euclidean theorem.) Sumer 6, 39–54 (2 plates) (1950).

This Old-Babylonian text, excavated near Baghdad, deals with a right triangle and similar triangles formed by constructing successive altitudes on the hypothenuse. The translation and commentary suffers from an incorrect use of the zero symbol in writing numbers in the sexagesimal system. *O. Neugebauer* (Providence, R. I.).

Yanovskaya, S. A. On the theory of Egyptian fractions. Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya 1, 269–282 (1947). (Russian)

Zubov, V. P. From the history of atomistics in the middle ages. Akad. Nauk SSSR. Trudy Inst. Istorii Estestvoznaniya 1, 283–314 (1947). (Russian)

Vernet, J., and de Orús, J. J. The transformation of astronomical coordinates among the Arabs. Gaceta Mat. (1) 2, 78–82 (1950). (Spanish)

Cassina, Ugo. Sull'origine ed evoluzione storica della geometria. Period. Mat. (4) 28, 1–12, 73–84 (1950).

Fleckenstein, J. O. The line of descent of the infinitesimal calculus in the history of ideas. Arch. Internat. Hist. Sci. (N.S.) 3, 542–554 (1950).

Dijksterhuis, E. J. Works on history of science published in the Netherlands in the years 1930–1947. Scripta Math. 16, 43–59 (1950).

Artobolevskiĭ, I. I., and Levitskiĭ, N. I. P. L. Čebyšev and the Russian theory of mechanisms. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 2, 34–52 (1947). (Russian)

After a brief scientific biography of Čebyšev, his fifteen papers on mechanisms are reviewed, with emphasis on their theoretical and practical importance. The originality and brilliance of this curiously neglected work is well argued.

A. W. Wundheiler (Chicago, Ill.).

Sadovskiĭ, L. E. Topics from the history of the development of mechanized mathematics in Russia. Uspehi Matem. Nauk (N.S.) 5, no. 2(36), 57–71 (1950). (Russian)

The first half of this paper is devoted to the work of Russian inventors in connection with early digital computing machines. Attention is given principally to P. L. Čebyšev who found (1881) a satisfactory solution to the problem of carry-over and developed a machine in which multiplication and division were automatic, thus "by many years anticipating the essential arrangement of the mechanism for automatic multiplication in contemporary calculating machines, the difference consisting only in details of construction." The machine patented in 1878 by V. T. Odner and thereafter widely distributed also receives attention. In the latter part of the paper two planimeters, patented in 1854 and 1856 by P. A. Zarubin, are described in some detail. These make use of the disc and friction wheel like pre-Amsler (1856) instruments such as that of T. Gonnella (1825) [O. Henrici, Rep. British Assoc. Advance. Sci. 1894, 496–523]. Reference is of course made to the early differential analyzer designed by and built (1904–1911) for A. N. Krylov as a result of ideas suggested by the work of W. Thomson [Proc. Roy. Soc. London. 24, 266–275 (1876)].

R. Church (Annapolis, Md.).

Titchmarsh, E. C. Obituary: Godfrey Harold Hardy. J. London Math. Soc. 25, 82–101 (1950).

Part 2 of vol. 25 of this journal is devoted entirely to a discussion of Hardy and his mathematical work [see the following review]. The obituary is followed by a list of Hardy's published papers.

Some aspects of Hardy's mathematical work. J. London Math. Soc 25, 102–138 (1950).

This discussion of Hardy's work consists of the following essays: (1) "Early work on divergent series" by L. S. Bosanquet; (2) "Integral equations" by F. Smithies; (3) "Diophantine approximation" by L. J. Mordell; (4) "The additive theory of numbers" by A. E. Ingham; (5) "Waring's problem" by H. Davenport; (6) "The Riemann zeta-function, and lattice-point problems" by E. C. Titchmarsh; (7) "Inequalities" by R. Rado; and (8) "Fourier series" by A. C. Offord.

FOUNDATIONS

Wang, Hao. Existence of classes and value specification of variables. J. Symbolic Logic **15**, 103–112 (1950).

For the introduction of classes defined by abstraction, the author suggests replacing D9 of Quine [Mathematical Logic, Norton, New York, 1940; these Rev. **2**, 65] by the following rule: $\ulcorner(\beta\varepsilon\hat{\alpha}(\varphi))\urcorner$ is an abbreviation for $\ulcorner((\exists\gamma)(\beta\varepsilon\gamma).\varphi[\alpha,\beta])\urcorner$, where γ is distinct from β, and where "$\varphi[\alpha,\xi]$" abbreviates "the expression ψ which is like φ' except for containing free occurrences of ξ whenever φ' contains free occurrences of α, φ' being an alphabetic variant of φ for which there exists such a ψ." The author shows an equivalence of his definition to that of Quine, and argues that the new treatment is closer to the traditional intuitive one. The remainder of the paper is concerned with contextual definition of variables having as range a limited class of classes. The author shows that for every theorem for all classes there is a corresponding one for those classes of the limited range, likewise that the theorems and principles of the first order predicate calculus remain true upon replacing variables by variables over the limited range provided the latter is not empty. *D. Nelson* (Washington, D. C.).

Henkin, Leon. Completeness in the theory of types. J. Symbolic Logic **15**, 81–91 (1950).

This investigation concerns a formal system $\mathfrak{F}$ given by Church [same J. **5**, 56–68 (1940); these Rev. **1**, 321]; the methods are applicable to many other systems which contain either a functional calculus of higher order or a theory of types. The type symbols in the system considered here are introduced as follows: 0 is the type of propositions; 1, that of individuals; $(\alpha\beta)$, that of functions whose arguments have type β and whose values have type α. By a general model M is meant a family of domains, one for each type symbol, such that D_0 is the set of two truth values T and F, D_1 is an arbitrary set of individuals, and $D_{\alpha\beta}$ is some class of functions defined over D_β with values in D_α. If $D_{\alpha\beta}$ is always the class of all these functions, M is a standard model. The notion of completeness of the formal system can be defined with respect to standard models only or to all general models. While $\mathfrak{F}$ is incomplete in the standard sense, it is here shown to be complete in the general sense. The proof results from the following theorem I. If Λ is any consistent set of cwff's (cwff = well formed formula without free variables), there is a general model (in which each domain D_α is denumerable) with respect to which Λ is satisfiable. To prove this, Λ is first extended to a maximal consistent set Γ, simply by enumerating the expressions of $\mathfrak{F}$ and adjoining successively all those formulas which are consistent with Λ and the previously adjoined formulas. Then the set of cwff's of type α is partitioned in equivalence classes mod Γ; two formulas are in the same class if the formula expressing their equivalence is deducible from Γ. Taking for D_1 the class of the equivalence classes of individuals, the author constructs by induction with respect to α a general model for which Γ is satisfiable. Now, to prove the completeness theorem, let A_0 be any wff and B_0 the cwff resulting from A_0 by binding all its variables by initial general quantifiers. If A_0 is valid in every general model, $\sim B_0$ is F in every general model; thus by theorem I, $\sim B_0$ is inconsistent; then A_0 is deducible. As a further consequence of theorem I we have: a set Γ of cwff's is satisfiable with respect to some model of denumerable domains D_α if and only if every finite subset of Γ is satisfiable. This theorem yields a simple method for constructing nonstandard models for the Peano axioms, the possibility of which was first proved by Skolem [Norsk Mat. Forenings Skr. **2**, no. 10, 73–82 (1933)]. Remarks. (1) Results very closely connected to those of this paper were deduced by Mostowski [same J. **12**, 33–42 (1947); these Rev. **10**, 93]. (2) As, since Hilbert, finitist (intuitionist) methods have special rights in metamathematics, it should be mentioned in a paper like this, that the results are classically, not intuitionistically, valid. *A. Heyting* (Amsterdam).

Quine, W. V. On decidability and completeness. Synthèse **7**, 441–446 (1949).

This is a paper on nonintuitionistic metamathematics. Let C be a recursive class of expressions; it is supposed that each member of C is either true or false. A complete proof procedure for C consists in a recursive binary relation R which has the set of true expressions in C as its converse domain; xRy may be interpreted as "x is a proof for y," and "y is true" is equivalent to $(Ex)(xRy)$. Similarly, a complete disproof procedure consists in a recursive relation S such that "y is true" is equivalent to $(x)(xSy)$, where xSy may be interpreted as "x is not a disproof of y." By a theorem of Kleene [Trans. Amer. Math. Soc. **53**, 41–73 (1943), p. 56; these Rev. **4**, 126] there follows: if an effective class of statements admits both a complete proof procedure and a complete disproof procedure, it admits a decision procedure, i.e., the class of its true expressions is recursive. This theorem applies also to classes of expressions which contain free variables. For a class of statements such that all negations of members are members further results are obtained by the remark that for such a class a complete proof procedure gives a complete disproof procedure and conversely. These results do not apply to classes of expressions with free variables. The author illustrates by examples, utilizing the incompleteness theorems of Gödel and Church, how an extension of a system for which a decision procedure exists may not admit a complete proof procedure or a complete disproof procedure. He considers also the case of a schematization of a class C of statements, i.e., a class C' of expressions containing free variables, such that a member of C appears when constants in a given class K are substituted for the free variables. *A. Heyting* (Amsterdam).

Quine, W. V. On natural deduction. J. Symbolic Logic **15**, 93–102 (1950).

Um die Abweichungen des klassischen Prädikatenkalküls (1. Stufe) vom umgangssprachlichen Schliessen zu verringern, schlägt Verf. eine Verbesserung das sogenannten natürlichen Systems von Gentzen [Math. Z. **39**, 176–210, 405–431 (1934)] vor. Die Individuenvariablen werden in einer Abzählung vorgegeben; und z.B. darf von $\varphi(\beta)$, bzw. $\exists_\alpha\varphi(\alpha)$, auf $\forall_\alpha\varphi(\alpha)$, bzw. $\varphi(\beta)$, nur dann geschlossen werden, wenn β höheren Index als die in $\forall_\alpha\varphi(\alpha)$, bzw. $\exists_\alpha\varphi(\alpha)$, vorkommenden freien Variablen hat. Hinter der erschlossenen Formel muss β angezeigt werden ("flagged") und darf dann nicht nochmal auf diese Weise verwendet werden. Eine Deduktion ist erst dann fertig ("finished"), wenn im Ergebnis keine "flagged" Variablen vorkommen. Die übrigen Regeln unterscheiden sich nur unwesentlich von den Gentzenschen. Es wird bewiesen, dass sich mit diesem System dieselben Formeln ableiten lassen, wie mit dem klassischen.

P. Lorenzen (Bonn).

Kleene, S. C. A symmetric form of Gödel's theorem. Nederl. Akad. Wetensch., Proc. **53**, 800–802 = Indagationes Math. **12**, 244–246 (1950).

The author exhibits two disjoint recursively enumerable sets (of natural numbers) C_0 and C_1 which cannot be separated by a general recursive set. He shows, in fact, that, given any two disjoint recursively enumerable sets D_0 and D_1 such that $C_0 \subset D_0$ and $C_1 \subset D_1$, there can always be found a number f not in $D_0 + D_1$. This proves that there is no exact parallelism between the theory of recursively enumerable and recursive sets on the one hand, and that of analytic and Borel sets on the other. *H. B. Curry* (Louvain).

Ridder, J. Formalistische Betrachtungen über intuitionistische und verwandte logische Systeme. II. Nederl. Akad. Wetensch., Proc. **53**, 446–455 = Indagationes Math. **12**, 98–107 (1950).

The author continues his program for the characterization of various subsystems of classical logic using two constants to be interpreted respectively as a true and a false statement [Nederl. Akad. Wetensch., Proc. **53**, 327–336 = Indagationes Math. **12**, 75–84 (1950); these Rev. **11**, 636]. In the present paper axioms are given for the systems corresponding to classical and intuitionistic propositional calculi as well as the minimal calculus of Johansson. Systems dual to these are also described (the classical system is self dual). Independence of axioms is discussed, duality relations are demonstrated, and the extension to predicate calculi of first order is presented. *D. Nelson* (Washington, D. C.).

Ridder, J. Formalistische Betrachtungen über intuitionistische und verwandte logische Systeme. III. Nederl. Akad. Wetensch., Proc. **53**, 787–799 = Indagationes Math. **12**, 231–243 (1950).

In this paper, a continuation of the paper reviewed above, calculi of the Gentzen type [Math. Z. **39**, 176–210, 405–431 (1934)] are presented for four of the eleven systems described in the preceding papers, namely for the system corresponding to the positive definite calculus, for a system containing an alternation operation, which is intermediate between the positive definite and minimal calculi, and for the respective duals of these two systems. *D. Nelson.*

McKinsey, J. C. C. A new definition of truth. Synthèse **7**, 428–433 (1949).

This is a semantical definition of truth of the same general nature as that of Tarski [Studia Philos. **1**, 261–405 (1935)]. It differs from Tarski's in two principal respects. In the first place it does not use the notion of satisfaction of a propositional function by a set of entities. On the other hand it depends more heavily on intuitive set theory. The definition is given explicitly only for a very simple object language. *H. B. Curry* (Louvain).

Alves, Maria Teodora. Hauber's law demonstrated by Boolean algebra. Gaz. Mat., Lisboa **10**, no. 41–42, 17–19 (1949). (Portuguese)

Hauber's law is a statement concerning reversibility of certain proofs involving mutually exclusive and exhaustive cases. It is a triviality from the standpoint of modern logic. The author gives a few examples of its use as illustrations of the utility of logic. Her formal developments are made unintelligible by a confusion between conjunction and alternation. *H. B. Curry* (Louvain).

van Melsen, A. G. M. The nature of logic. Synthèse **7**, 434–440 (1949).

This is a philosophical paper. The thesis is that logic cannot be completely separated from philosophy.

H. B. Curry (Louvain).

***Beth, E. W. Les fondements logiques des mathématiques.** Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1950. 222 pp. 1400 francs.

The author presents a brief but general introduction to problems of the foundations of mathematics. Sufficient technical detail is included on many topics to give a good understanding of methods. At all times the author endeavors to indicate divergent viewpoints and to direct the reader to literature covering those matters not treated in detail. Initial and concluding chapters indicate the relation of foundational problems to various epistemological positions. The mathematical material begins with an informal discussion of the classical number system and includes a fairly extended account of Dedekind's approach to the definition of natural number. A precise formulation of the first order predicate calculus is followed by a discussion of the problem of a more general logic. In a chapter devoted to proof theory, consistency and completeness of the predicate calculus is established. Herbrand's theorem on the predicate calculus is outlined and the theorems of Gödel and Löwenheim-Skolem are stated as nonfinitary consequences. A decision procedure is carried out for a system which formalizes an order relation. In a chapter on logicism, Frege's sense-denotation distinction is described along with an informal account of his definition of natural number. A chapter on intuitionism explains and gives examples of Brouwer's concepts of choice-sequence and spread. A separate chapter devoted to syntax contains a brief account of the Gödel incompleteness theorems. There are further chapters on set theory (outlining the Zermelo-Fraenkel-Skolem theory), semantics, and the paradoxes. A list of exercises is appended. [The following typographical errors might impede the reader: p. 18, l. 3 from below, "c, b" for "$c\ b$"; p. 29, l. 22, insert "e" at end of line; p. 39, l. 2 from below, delete last "U"; p. 47, l. 9, "$\bar{U}$" for "U"; p. 57, l. 5, "$v(x, t(z))$" for "$v(x(t), z)$"; p. 57, l. 3 from below, "$y(x)$" for "$y(t)$"; p. 57, last line, "$y(x)$" for "$y(z)$"; p. 65, l. 3 from below, "D" for "T"; p. 129, l. 11, "$z\varepsilon y$" for "$z\ y$"; p. 129, l. 2 from below, insert "ε" at end of line; p. 141, l. 4, delete first "2"; p. 163, l. 13, delete "$=$."] *D. Nelson* (Washington, D. C.).

***Maziarz, Edward A. The Philosophy of Mathematics.** Philosophical Library, New York, N. Y., 1950. viii+286 pp. $4.00.

This book concerns the philosophical nature of mathematical abstraction. The introduction points out the persistent confusion among scientific, mathematical, logical, and metaphysical abstraction. Part I is a history of the views that have been taken toward the nature of mathematical abstraction and its philosophical role in science. Part II presents the view claimed to be correct.

Content. The author begins with the fundamental distinction between a mathematical (or logical) system as a method of understanding, and this system as an object to be understood. He emphasizes the consequences of confusing the two aspects, and takes as his basic question, what system (empirical, rational, or otherwise) may be used to appraise mathematics (object)? He denies the competence of mathematics (method) and even metamathematics (p. 15), and

asserts that only metaphysics is suitable (p. 119). He concludes that mathematics is distinct from logic and other sciences, and that quantity is the ultimate subject matter of mathematics (p. 193).

Style. This book is largely a collection of citations and digests from the literature. Thus it provides a historical survey, and it includes an extensive bibliography.

In part I, each main division of chapters III to VI is titled with the name of the man whose concepts are being presented, e.g., Aristotle, Leibniz, Kant. In part II, the concepts presented are not so titled. [In part II, there are at least 26 numbered references to Aristotle, and at least 59 numbered references to St. Thomas Aquinas, but only about 5 numbered references to all other philosophers (such as Leibniz) earlier than roughly 1900. Moreover, the author states, for example, in reference 70 on page 181, "The question of the specification of sciences outlined above, adheres closely to the thought of Saint Thomas Aquinas."]

The presentation seems dogmatic. For example, on p. 148 the author writes, "True philosophy has always argued with idealists and sceptics and maintained that our minds know things and not merely impressions or ideas of things." Little effort is made to present new evidence or to meet criticisms. In fact, the author says on p. 164, "Any questioning of the validity of these principles, or any attempt to validate their use, while a frequent indulgence of scientists in this realm, is properly a metaphysical view outside the scope of these first sciences." [Cf. R. A. Kocourek, "Introduction to the Philosophy of Nature" (a study of St. Thomas), caption to Lesson II, p. 32: "The opinions of the ancient philosophers about the principles of nature and of beings. It does not pertain to natural science to disprove the opinions of some of these."] Moreover, the author gives little indication of the basis, method, structure, or mode of application of metaphysics other than as a "judgment of separation" (pp. 159, 168).

Reviewer's criticisms. Many statements in this book seem to contradict well-established experience. For example: "Both requisites [for scientific knowledge], immateriality and immobility or removal from matter and motion, are achieved by the mind . . ." (p. 159). This seems to contradict the fundamental principles of relativity (such as variation of mass with velocity). Again, "Quantity, being an accident of bodies, cannot exist or be conceived apart from some reference to matter. In mathematical abstraction I abstract quantified substance" (p. 165). This seems to contradict the possibility of quantity of energy. [It is very doubtful if St. Thomas (died 1274) had any clear concept of energy, for as late as the time of Leibniz (17th century) the general concept of energy was only beginning to be formulated in any precise form.] Again, "it is only by knowing the separation of a being from other beings that I am able to discern that being is" (p. 168). This seems to contradict the fundamental principles of quantum mechanics (such as the interaction of electrons).

The author attempts to avoid a narrow interpretation of the principle that "quantity . . . [is] the ultimate subject matter of mathematical demonstration" in order to encompass recent mathematical developments (p. 193). He says (p. 193): "The notion of quantity . . . means the order of the parts of substance," and without defining this last expression, he gives it a very broad interpretation. However, the quotation comes from St. Thomas Aquinas, "In Librum Boethii De Trinitate Quaestiones Quinta et Sexta," Quaestio V, Art. 3 ad 3, where it is stated that "Quod quidem quantitati competit, in quantum in sui ratione situm quasi differentiam constitutivam habet, quod nihil est aliud quam ordo partium." According to the translation by Sister Rose E. Brennan, this may be rendered as follows: "And this belongs to quantity inasmuch as by its nature it has location as constitutive difference, which is none other than the order of parts." It is difficult to see how the author's more general interpretation may be justified.

Reviewer's conclusion. De mortuis, non ex mortuis, nihil nisi bonum. *C. C. Torrance* (Annapolis, Md.).

Koksma, J. Concerning the function concept. Euclides, Groningen **25**, 313–340 (1950). (Dutch)

Lévy, Paul. Axiome de Zermelo et nombres transfinis. Ann. Sci. École Norm. Sup. (3) **67**, 15–49 (1950).

This somewhat polemical article consists of three parts. The first contains an attempt to show that, although Zermelo's axiom has some paradoxical consequences, the rejection of this axiom leads to results that are even more paradoxical and make it almost necessary to doubt human reason. The second part is devoted to Richard's antinomy and to what the author considers the essential role which certain transfinite ordinals play in its explanation. The last part is concerned with a method, due to A. Denjoy [e.g., L'Énumération Transfinie . . ., Gauthier-Villars, Paris, 1946, pp. 151–167, 203–206; these Rev. **8**, 254], of defining any ordinal number of the second class by means of an ordinary infinite sequence of natural numbers suitably chosen from a certain set K of such sequences, and without the intervention of transfinite recurrence. It appears to the author that if one attempts to set up a one-to-one correspondence between the elements of K and the points of a line segment in a manner which is practically utilizable and which avoids certain disadvantages of Cantor's indirect method of establishing such a correspondence, then one necessarily runs again into all the difficulties of the transfinite which Denjoy's result intends to avoid.

F. Bagemihl (Rochester, N. Y.).

Dingle, Herbert. A theory of measurement. British J. Philos. Sci. **1**, 5–26 (1950).

The author outlines the application of operationism to the development of a theory of measurement. He argues against the existence of objective absolute properties, but ignores subjective relative aspects and the ultra-fundamental psychometric measurements based on them. The claim is made that operations constitute the basic elements of physical theory, but no indication is given of the underlying logic of operations. A measurement is defined to be any precisely specified operation that yields a number. This definition implies the freedom of the investigator to perform any operation he may find possible and useful, and an operation is precisely specified when those conditions are specified that are necessary to give results that stand in simple relations to others. It is considered that repetition and averaging are essential parts of measurement, and that the tossing of a coin n times is in no respect different from a recognised measurement such as the measurement of a length. The question as to why both of these operations give consistent results in the long run is regarded as not a logical problem, but an elementary emotion. The interpretation of the indeterminacies of quantum mechanics is reduced, by denying the existence of properties, to the usual uncertainties of measurements in general. However, this last point seems to

emphasize the contention that empirical theories do not resolve, but merely ignore, the problems of reality.

C. C. Torrance (Annapolis, Md.).

Destouches-Février, P. Logique et théories physiques. Synthèse 7, 400–410 (1949).

This expository article outlines a methodology whereby the results of, and predictions from, experimental measurements can be expressed in terms of operations in a linear space Y. It is shown, from the properties of Y, that quantum logic cannot be finite valued.

C. C. Torrance (Annapolis, Md.).

Destouches, J. L. Intervention d'une logique de modalité dans une théorie physique. Synthèse 7, 411–417 (1949).

This paper outlines a general theory for unifying two or more incompatible physical theories by weakening the logic used to construct the englobing theory.

C. C. Torrance (Annapolis, Md.).

Whitrow, G. J. On the foundations of dynamics. British J. Philos. Sci. 1, 92–107 (1950).

Popper, Karl R. Indeterminism in quantum physics and in classical physics. British J. Philos. Sci. 1, 117–133 (1950).

ALGEBRA

Usai, Giuseppe. Concordanze nelle permutazioni. Atti Accad. Gioenia Catania (6) 6 (1943/49), no. 13, 8 pp. (1950).

Soit $N^{(\alpha)}$ le nombre des permutations des n éléments $1, 2, \cdots, n$ qui présentent α "concordanze," c'est-à-dire dans lesquelles α éléments sont à leur place naturelle (problème des rencontres [cf. Euler, Opera, v. 7, pp. 542–545]). La valeur de n étant fixée, l'auteur cherche pour quelle valeur de α la fonction $N^{(\alpha)}$ est maximum. Il trouve que le maximum est atteint pour $\alpha=0$ si $n=2k$ et pour $\alpha=1$ si $n=2k+1$ ($k>1$). Les démonstrations sont élémentaires et particulièrement claires.

A. Sade (Marseille).

Radhakrishna Rao, C. The theory of fractional replication in factorial experiments. Sankhyā 10, 81–86 (1950).

The author has previously [Suppl. J. Roy. Statist. Soc. 9, 128–139 (1947); these Rev. 9, 264] shown that from an array of strength $(d+k-1)$ all the main effects and interactions up to order $k-1$ can be estimated in a simple manner when interactions of order greater than $d-1$ are absent. The estimates are moreover orthogonal to each other. This fact is utilized for the construction of factorial designs with fractional replications from which main effects and first order interactions can be easily estimated. The method is applicable only when the number of levels of each factor is a prime power.

H. B. Mann (Columbus, Ohio).

Banerjee, K. S. A note on the fractional replication of factorial arrangements. Sankhyā 10, 87–94 (1950).

Finney [Ann. Eugenics 12, 291–301 (1945); these Rev. 7, 213] introduced factorial experiments with a fractional number of replications. However, for the 2^n factorial experiment Finney only considered $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ replicates. The author discusses a $\frac{3}{4}$ replicate, that is to say, experiments with only one-fourth of the treatment combination omitted. The analysis of such designs is outlined by the author and illustrated by an example. Although more complete than the analysis of the $(\frac{1}{2})^n$ replicate it does not present any insurmountable computational difficulties. Methods of confounding are also discussed.

H. B. Mann.

Bhattacharyya, K. N. Problems in partially balanced incomplete block designs. Calcutta Statist. Assoc. Bull. 2, 177–182 (1950).

Partially balanced incomplete block designs were introduced and defined by R. C. Bose and K. R. Nair [Sankhyā 4, 337–372 (1939)]. The author states without proof four theorems in partially balanced incomplete block designs. The first of these is a nonexistence theorem. The other three are constructive. The author's second theorem leads to a solution of the design $v=17$, $b=34$, $r=8$, $k=y$, $n_1=n_2=8$, $\lambda_1=1$, $\lambda_2=2$. A complete list of designs with $r\leqq 10$, constructed by means of the author's theorems, will be published later.

H. B. Mann (Columbus, Ohio).

Varoli, Giuseppe. Identità numeriche. Boll. Un. Mat. Ital. (3) 4, 250–254 (1949).

The author considers some special cases of results of Sibirani [Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) 5 (1947/48), 27–34 (1949); these Rev. 12, 86] and shows that these examples lead to certain identities. The identities are too involved to be quoted here; they are elementary and effect the transformation of a finite sum of powers and binomial coefficients into another similar finite sum. No remarks are offered about the significance or application of these identities.

A. Erdélyi (Pasadena, Calif.).

***Kuroš, A. G. Kurs vysšeĭ algebry. [A Course of Higher Algebra].** 2d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 335 pp.

This is an elementary text on polynomials and matrices. Noteworthy features: (1) unusually detailed explanations and examples, (2) the inclusion in chapter 8 of the theorems of Sturm, Budan, and Hurwitz, and two proofs of the fundamental theorem of algebra, (3) an appendix describing the achievements of Russian algebraists. The chapter headings are as follows. I. Fields, complex numbers. II. Determinants. III. Matrix algebra. IV. Systems of linear equations. V. Quadratic forms. VI. Polynomials over an arbitrary field. VII. Polynomials in several variables. VIII. Polynomials with real or complex coefficients. IX. Polynomials with rational coefficients. X. Groups, algebras.

I. Kaplansky (Chicago, Ill.).

***Zurmühl, Rudolf. Matrizen. Eine Darstellung für Ingenieure.** Springer Verlag, Berlin-Göttingen-Heidelberg, 1950. xv+427 pp. 25.50 DM.

Although this book is primarily written for engineers it contains useful information of general interest; it includes many examples worked out in detail. The first three chapters deal with the elementary material concerning matrices, determinants, quadratic, and bilinear forms. Chapter 4 introduces the characteristic value problem, first for real symmetric, then for arbitrary real matrices, and finally in the generalized form: $Ax=\lambda Bx$. Matrices with complex elements are treated next. Chapter 5, "Structure of matrices," deals with the Cayley-Hamilton theorem, elementary divisors, and normal forms. Orthogonality theorems

for the eigenvectors of matrices and their transposes are studied in detail. The elementary divisors are discussed following the treatment of H. Wielandt. Next, functions of matrices, in particular power series, and equations for matrices are considered. Chapter 6 deals with numerical methods. Gauss' elimination process for the solution of linear systems of equations and variations on it due to Cholesky and Banachiewicz are discussed. Among iteration processes those of Gauss-Seidel and Southwell (relaxation) are considered. The latter is traced back by the author to a lecture of Gauss edited by Dedekind [see Dedekind, Werke, vol. 2, pp. 293–306] which also described a not generally known method for improving the convergence of the process. The iteration procedure for the determination of the dominant root, and Collatz' inclusion theorem which is a generalization of it, are discussed. The case of nonlinear elementary divisors is treated using methods of Wielandt and Duncan and Collar. Various methods for the determination of all roots are discussed, including the method of reduction to matrices of smaller dimension and various methods for the determination of the characteristic polynomial. Chapter 7 deals with applications. *O. Todd-Taussky* (Washington, D. C.).

Robinson, G. de B. Induced representations and invariants. Canadian J. Math. **2**, 334–343 (1950).

The problem of expressing an invariant matrix of an invariant matrix as a direct sum of invariant matrices is of importance in invariant theory. The problem has been studied previously by S-functional analysis and the equivalent of the solution is given by the expansion of the "new multiplication" or "plethysm" of S-functions, $\{\lambda\}\otimes\{\mu\}$, in terms of S-functions. Previously, methods of computing this expansion have been either laborious or else tentative in nature. The author describes a step-by-step procedure which makes a very definite contribution to the problem. The method employed is not that of S-functional analysis, but the parallel theory concerning the representations of the corresponding imprimitive subgroups of the symmetric groups. The subgroup corresponding to $\{\lambda\}\otimes\{\mu\}$ is denoted by $[\lambda]\odot[\mu]$. The first principal result is that if

$$[\alpha]\odot[\beta_i]=\sum\rho_{ii}[\lambda_i],\quad [\beta_i]\cdot[1]=\sum e_{ki}[\beta_k'],$$
$$[\lambda_i]\cdot[\alpha]=\sum\sigma_{ji}[\lambda_j'],\quad [\alpha]\odot[\beta_k']=\sum\rho_{jk}'[\lambda_j'],$$

then $\sum\rho_{jk}'e_{ki}=\sum\sigma_{ji}\rho_{ii}$.

If the expansions for $[\alpha]\odot[\beta]$ are known for all partitions (β) of n, the formula indicates a method of deducing the expansions of $[\alpha]\odot[\beta']$ for partitions (β') of $(n+1)$, by a procedure which the author likens to integration. This formula, while formally new, is not essentially new since its equivalent can be deduced by S-functional analysis from the distributive properties of the operation $\otimes$ with respect to addition and multiplication, and the ambiguities in the "integration" procedure are the same ambiguities that have occurred in the S-functional methods. However, the author next proceeds to obtain certain "boundary conditions." He uses the theory of hooks and star diagrams to assign a suffix $+$, $-$, or 0 to the partition of each representation. The integration procedure described above must satisfy certain restrictions with respect to these suffixes. The author claims that these restrictions are sufficient to remove completely all ambiguity in the integration procedure in every case. Such examples as have been worked out by the reviewer do in fact lead to solutions without ambiguity. A reasonable method would therefore seem to be achieved for the step-by-step evaluation of $[\alpha]\odot[\beta]$ or $\{\alpha\}\otimes\{\beta\}$ for all cases.

D. E. Littlewood (Bangor).

Todd, J. A. Note on a paper by Robinson. Canadian J. Math. **2**, 331–333 (1950).

In a recent paper Robinson [same J. **1**, 166–175 (1949); these Rev. **10**, 504] obtained an explicit formula for an invariant matrix of an invariant matrix as a direct sum of invariant matrices. The author obtains an equivalent result by the use of S-functional analysis. Use is made of the "new multiplication," or as it has more recently been termed the "plethysm" of S-functions, as defined by the reviewer [Philos. Trans. Roy. Soc. London. Ser. A. **239**, 305–365 (1944); these Rev. **6**, 41]. The author quotes the formula $m!\{\alpha\}\otimes S_k=\sum_\rho h_\rho\chi_\rho^{(\alpha)}S_{[k\rho]}$ from an earlier paper [Proc. Cambridge Philos. Soc. **45**, 328–334 (1949); these Rev. **10**, 672] and uses it to prove

$$(m!)^n n!\{\alpha\}\otimes\{\beta\}=\sum_{\rho,\rho_1,\cdots,\rho_n} h_\rho\chi_\sigma^{(\beta)}\left(\prod_{i=1}^n h_{\rho_i}\chi_{\rho_i}^{(\alpha)}\right)\chi_{[n,\sigma]}^{(\lambda)}\{\lambda\}.$$

He shows that this is equivalent to the result obtained in Robinson's paper. *D. E. Littlewood* (Bangor).

Parker, W. V. On matrices whose characteristic equations are identical. Proc. Amer. Math. Soc. **1**, 464–466 (1950).

The following three theorems are proved by making suitable equivalence or similarity reductions on the matrices involved. (1) If $ACA=kA$, k a scalar, then

$$\det\|xI_n-AB\|=x^{n-r}\varphi(x)$$

and

$$\det\|xI_n-A(B+C)\|=x^{n-r}\varphi(x-k),$$

where $r=\operatorname{rank}A$ and B is an arbitrary rectangular matrix of the same size as C. (2) If M and N are square and NM (or MN) $=N^2=0$, then M and $M+N$ have the same characteristic equation. (3) If A and B are square, S and T exist such that $SA=S^2=AT=T^2=0$ and $AB-S$ is similar to $BA-T$. *W. Givens* (Knoxville, Tenn.).

Parodi, Maurice. Un critère d'irréductibilité sur le corps des nombres rationnels du polynome caractéristique de matrices dont les éléments sont des entiers. C. R. Acad. Sci. Paris **231**, 97–99 (1950).

A result of A. Brauer [Duke Math. J. **13**, 387–395 (1946); these Rev. **8**, 192] concerning the characteristic roots of matrices is used to obtain sufficient conditions for the irreducibility in the field of rational numbers of the characteristic polynomials of matrices with integral rational coefficients. The result contains a criterion of Perron.

R. Brauer (Ann Arbor, Mich.).

Roth, William E. On the eliminant of $f(\lambda x)$ and $f(\lambda/x)$. J. London Math. Soc. **25**, 232–236, 354 (1950).

The author gives a proof using matrix methods of the following theorem stated and proved by analytic means by E. B. Elliott [Proc. London Math. Soc. (1) **25**, 173–184 (1894)]. The Sylvester dialectic eliminant of the polynomials $f(\lambda x)$ and $x^n f(\lambda/x)$ is the product of $f(\lambda)$, $f(-\lambda)$ and a perfect square factor. The square factor is explicitly given.

B. W. Jones (Boulder, Colo.).

Abstract Algebra

Wendelin, H. Zwei Verästelungssätze. J. Reine Angew. Math. **187**, 231–233 (1950).

Consider a system T of elements $M\equiv M_0, M_1, \cdots, M_{r_0}, M_{11}, \cdots, M_{i_1\cdots i_n}, \cdots$; $i_n\equiv\{1, 2, \cdots, r_{i_1\cdots i_{n-1}}\}$. Theorem. Of

the following conditions, A implies B and B is equivalent to C. A: For infinitely many lengths of subscript there are E-objects (objects with a given property E); and if $M_{i_1\cdots i_n}$ is an E-object so is $M_{i_1\cdots i_{n-1}}$. B: In T there is an infinite sequence of E-objects M, M_{i_1}, $M_{i_1 i_2}$, $\cdots$. C: In T there are finite sequences of E-objects M, M_{i_1}, $M_{i_1 i_2}$, $\cdots$, $M_{i_1 i_2\cdots i_n}$ of each length $n=0, 1, 2, \cdots$. Some applications to topology are mentioned. *P. M. Whitman* (Silver Spring, Md.).

Iseki, Kiyoski. Une condition pour qu'un lattice soit distributif. C. R. Acad. Sci. Paris **230**, 1726–1727 (1950).

The author shows that, if in a lattice L every irreducible ideal is prime, then L is distributive (the converse being also true). This can be generalized as follows. If every completely irreducible ideal is prime, L has to be distributive.

L. Nachbin (Rio de Janeiro).

***Kuo, Ke-Chan. The imbedding problem for systems with an incomplete, commutative addition.** Abstract of a Thesis, University of Illinois, Urbana, Ill., 1950. ii+5 pp.

An add A is a system of elements with an incomplete composition, called addition. That is, for some $a, b \in A$, $a+b$ is defined, and is unique whenever it is defined. If whenever $a+b$ is defined, then $b+a$ is also defined and $a+b=b+a$, the add is commutative. The author is concerned with conditions under which it is possible to imbed a commutative add into a group. The procedure used involves an analysis of vectors with elements in A, and such ideas as similarity of such vectors as defined by Baer [Amer. J. Math. **71**, 706–742 (1949); these Rev. **11**, 78], and other related concepts. No proofs are given. *N. H. McCoy.*

Evans, T. A note on the associative law. J. London Math. Soc. **25**, 196–201 (1950).

Let G be a groupoid. Let P_i, Q_i $(i=1, 2, 3, 4, 5)$ be mappings of G into itself. A law of the form

$$\text{(A)} \qquad (xP_1\cdot(yP_2\cdot zP_3)P_4)P_5=((xQ_1\cdot yQ_2)Q_3\cdot zQ_4)Q_5$$

for all x, y, z in G is called a generalized associative law. The following theorems are proved. A finite groupoid with a unit, and satisfying a law of the form (A), where the P_i are permutations, is associative. A groupoid with a unit, division on one side, and satisfying a law of the form (A), where the P_i and Q_i are permutations, is a group. The property of possessing a law of the form (A) is an isotopic invariant for groupoids. A quasigroup is isotopic to a group if and only if it satisfies a law of the form (A), where the P_i and Q_i are permutations. *F. Kiokemeister.*

Jenner, W. E. The radical of a non-associative ring. Proc. Amer. Math. Soc. **1**, 348–351 (1950).

Let R be a (nonassociative) ring whose (associative) transformation ring A satisfies the minimum condition. There is a minimal ideal M_1 of R such that $R-M_1$ has no absolute divisors of zero. Consider the ideal M_1 for the ring $R-RN$ where N is the radical of A. The radical of R is defined to be the complete reciprocal image M of M_1 under the natural homomorphism $R\to R-RN$. Then M is the minimal ideal of R such that $R-M$ is semisimple. *R. D. Schafer.*

Jacobson, N. Some remarks on one-sided inverses. Proc. Amer. Math. Soc. **1**, 352–355 (1950).

Let $\mathfrak{A}$ be a ring containing an identity 1 and a pair of elements u, v such that $uv=1$. The author observes that the elements $e_{ij}=v^{i-1}u^{j-1}-v^iu^j$ (where $u^0=v^0=1$) satisfy the multiplication table for matrix units, and deduces: (1) if $\mathfrak{A}$ satisfies either the ascending or the descending chain condition for principal right ideals generated by idempotent elements, then $vu=1$; (2) if $vu\neq 1$, then $\mathfrak{A}$ contains a right ideal $(\sum e_{ii}\mathfrak{A})$ which is a direct sum of an infinite number of $\mathfrak{A}$-isomorphic right ideals; (3) if u has a second right inverse, then u has an infinity of right inverses (for $ue_{11}=0$ in the proof, read $ue_{1k}=0$); (4) if $vu\neq 1$ and $\mathfrak{A}$ is an algebra over a field Φ, then the subalgebra $\Phi(u, v)$ generated by u and v is primitive and has minimal one-sided ideals, and all such subalgebras corresponding to different pairs u, v are isomorphic. There are further remarks on the structure of $\Phi(u, v)$ generally and when u, v are matrices or linear transformations. Analogous theorems hold for the quasi-inverses of Andrunakievič [Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 129–178 (1948); these Rev. **9**, 564]. Reference is made to Baer [Bull. Amer. Math. Soc. **48**, 630–638 (1942); these Rev. **4**, 70], who obtained [loc. cit., p. 635] the conclusion of (1) under stricter chain conditions. Theorem (3) is attributed to Kaplansky (proof by structure theory, orally communicated).

I. M. H. Etherington (Edinburgh).

Asano, Keizo. Zur Arithmetik in Schiefringen. II. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **1**, 1–27 (1950).

The following notation is explained in the review of part I [Osaka Math. J. **1**, 98–134 (1949); these Rev. **11**, 154]. The only rings considered are those with an identity element and such that each element is either a unit or a divisor of zero. The paper is mainly concerned with simple rings and semisimple rings (finite direct sum of simple rings) having an order $\mathfrak{o}$ satisfying the (Noether) axioms (A_1) $\mathfrak{o}$ is a maximal regular order, (A_2) the ascending chain condition holds for the ideals contained in $\mathfrak{o}$, (A_3) each prime ideal is strongly divisorless. For a semisimple ring, these axioms are shown to be equivalent to a set of 5 axioms of Artin. It follows that the axioms (A_1)–(A_3) hold for any maximal regular order equivalent to $\mathfrak{o}$. Any ring having an order $\mathfrak{o}$ satisfying (A_1)–(A_3) is simple if and only if the double chain condition holds for the integral ideals containing a fixed nonzero element. The Deuring $\mathfrak{m}$-adic evaluation, definable in a simple ring, is discussed. *R. E. Johnson.*

Smiley, M. F. A remark on a theorem of Marshall Hall. Proc. Amer. Math. Soc. **1**, 342–343 (1950).

It is proved that the identity $x(yz-zy)^2=(yz-zy)^2x$ characterizes Cayley-Dickson algebras among alternative but not associative division rings. Albert's theorem that every alternative quadratic algebra is of finite dimension (necessarily 1, 2, 4, or 8) [Bull. Amer. Math. Soc. **55**, 763–768 (1949); these Rev. **11**, 76] is established for fields of characteristic two. [Reviewer's note. Bruck and Kleinfeld have subsequently announced that all alternative but not associative division rings are Cayley-Dickson algebras.]

R. D. Schafer (Philadelphia, Pa.).

Herstein, I. N. A proof of a conjecture of Vandiver. Proc. Amer. Math. Soc. **1**, 370–371 (1950).

The following results, generalising the well known theorem of Wedderburn that each finite division ring is commutative,

are proved: (1) a finite ring contains a divisor of zero not contained in its centrum, and (2) the same result with the weaker hypothesis, that each element of the ring generates a finite ring. *M. Krasner* (Paris).

Mařík, Jan. La réductibilité du déterminant ayant des indéterminées pour éléments, si l'on le considère comme un polynôme sur an anneau commutatif. Acta Fac. Nat. Univ. Carol., Prague no. 191 (1949), 11 pp. (1949).

Let R be a commutative ring with unit element 1, and x_{ij}, $i, j=1, 2, \cdots, n$, be indeterminates over R. The author considers the reducibility of the determinant $D=|x_{ij}|$ in the ring $P=R[x_{11}, x_{12}, \cdots, x_{nn}]$. Thus D is reducible if it can be expressed as the product of two elements of P, neither of which is a unit of P. If R is expressible as the direct sum of two rings, that is, if there exist nonzero idempotents j_1 and j_2 of R such that $j_1+j_2=1$, $j_1j_2=0$, then D is reducible since we can write $D=(j_1+j_2D)(j_2+j_1D)$. The principal result is that D is reducible if and only if R can be expressed as the direct sum of two rings. *N. H. McCoy.*

Pickert, Günter. Komposita transzendenter Körpererweiterungen. J. Reine Angew. Math. 187, 234–245 (1950).

The following result on composite fields is proved. Let k be a field, and K, K' be extensions of k contained in a common extension $\Re$; it is assumed that $\Re=KK'$, and that any transcendence base of K/k is also one of $\Re/K'$. If k is relatively algebraically closed in K and if K' is a simple extension (generated by adjunction of a single element) of a field which admits a separating transcendence base over k, then K and K' are linearly free in $\Re$.

This, and other results on the composition of fields, is used to derive certain properties of prime divisors of fields of algebraic functions of one variable. Let K'/k be a field of algebraic functions of one variable, and let the assumptions be as above. Let first $\mathfrak{p}$ be a variable prime divisor of $\Re$ (i.e., such that the elements $\neq 0$ of K' are of order 0 with respect to $\mathfrak{p}$). Assume that the direct product of the largest separable algebraic extensions of k in K and K' is a field. Then $\mathfrak{p}$ determines a "dependent" composite of K and K' (i.e., one which is algebraic over K) and every such composite of K and K' is obtained from exactly one variable prime divisor of $\Re/K$. Assume now that k is relatively algebraically closed in K; then there is a one-to-one correspondence between the fixed prime divisors $\mathfrak{p}$ of $\Re/K$ and the prime divisors $\mathfrak{p}^*$ of K', $\mathfrak{p}$ corresponding to $\mathfrak{p}^*$ if its valuation ring contains that of $\mathfrak{p}^*$. Moreover, if K has a separating base over k or if the residue field of $\mathfrak{p}^*$ is a simple extension of a separable extension of k [the published text says: "of a purely transcendental extension of k," but this seems to the reviewer to be a mistake], then it can be asserted that the residue field of $\mathfrak{p}$ is a composite of K and of the residue field of $\mathfrak{p}^*$. *C. Chevalley.*

Sikorski, Roman. A theorem on the structure of homomorphisms. Fund. Math. 36, 245–247 (1949).

Let $\mathfrak{S}$ be the set of all prime ideals of an arbitrary σ-complete Boolean algebra $\mathbf{A}$. In this paper classes $\mathbf{Z}$ and $\mathbf{N}$ of subsets of $\mathfrak{S}$ are defined so that the following statements are true: (a) $\mathbf{N}$ is a σ-ideal and $\mathbf{A}$ is isomorphic to $\mathbf{Z}/\mathbf{N}$; (b) if $\mathbf{A}_1$ is an arbitrary σ-complete Boolean algebra which is σ-homomorphic to $\mathbf{A}$, and if $\mathbf{Z}_1$ and $\mathbf{N}_1$ are defined for $\mathbf{A}_1$ in the same way as $\mathbf{Z}$ and $\mathbf{N}$ for $\mathbf{A}$, then the resulting homomorphism of $\mathbf{Z}/\mathbf{N}$ in $\mathbf{Z}_1/\mathbf{N}_1$ is induced by a mapping of the set $\mathfrak{S}_1$ of all prime ideals of $\mathbf{A}_1$ in the set $\mathfrak{S}$. (The term "induced" in statement (b) has the same meaning as in the author's related paper [same vol., 7–22 (1949); these Rev. 11, 166].) *H. M. Schaerf* (St. Louis, Mo.).

Theory of Groups

***Schubert, Jewell Emma. On groups of order 3^n and class 3.** Abstract of a Thesis, University of Illinois, Urbana, Ill., 1950. 10 pp.

A study is made of groups of order 3^n, generated by n generators u_i, all elements of which, except identity, are of order 3. Burnside [Quart. J. Math. Pure Appl. 33, 230–238 (1902)] has shown that such a group is finite, while F. Levi and van der Waerden [Abh. Math. Sem. Univ. Hamburg 9, 154–158 (1932)] has shown that such a group is of class 3 when no further relations exist. If we define $s_{ij}=u_i^{-1}u_j^{-1}u_iu_j$ and $t_{ijk}=s_{ij}^{-1}u_k^{-1}s_{ij}u_k$, then $s_{ji}=s_{ij}^{-1}$, $t_{jki}=t_{ijk}=t_{jik}^{-1}$. The $\binom{n}{2}$ commutators s_{ij} generate an Abelian commutator subgroup K, and the $\binom{n}{3}$ second commutators t_{ijk} generate the central C. The elements u_i, s_{ij}, t_{ijk} form a basis for the general group of order 3^e where $e=\binom{n}{1}+\binom{n}{2}+\binom{n}{3}$, and all other such groups having n generators are obtained as subgroups or quotient groups of this general group. The author lists all these groups for $n=4$, and discusses certain types for $n=5$ and $n=6$. An element of K is itself a commutator if and only if the exponents associated with the basis elements s_{ij} are a set of Plücker line coordinates. For $n=4$, each element t of C is a commutator of a noncommutator s of K with an element u of a uniquely determined coset of K in G. For $n=5$, some t's are obtainable in this way, but others are commutators of commutators, so there are two types of quotient group of index 3. For $n=6$ a third type of quotient group of order 3^{40} arises from elements of C that are not themselves commutators. *J. S. Frame.*

Neumann, B. H., and Neumann, Hanna. A remark on generalized free products. J. London Math. Soc. 25, 202–204 (1950).

H. Neumann [Amer. J. Math. 70, 590–625 (1948); these Rev. 10, 233] has proved a reduction theorem for the existence of free products with amalgamations. In the present note it is shown that the proof of this reduction theorem may be derived very easily from Schreier's theorem which asserts the existence of free products with one amalgamated subgroup. It is furthermore shown that similar considerations may be effected for direct products with amalgamations. *R. Baer* (Urbana, Ill.).

Baer, Reinhold. Ein Einbettungssatz für Gruppenerweiterungen. Arch. Math. 2, 178–185 (1950).

The main result of the paper has already been stated in another paper of the author [Math. Nachr. 2, 317–327 (1949); these Rev. 11, 496] and was mentioned in the review. It deals with the possibility of imbedding a set of extensions of a group M by a group E in a group extension H of a group N by E; the given extensions have to satisfy a certain natural condition. The group N can be chosen as a direct sum of M and a free Abelian group. The present paper contains the proof of this result. Further properties of the extension H constructed in the proof are given. As a special case, a theorem of Artin [cf. H. Zassenhaus, Lehrbuch der Gruppentheorie, Teubner, Leipzig-Berlin, 1937, theorem 22, p. 98] is obtained. *R. Brauer.*

Kaloujnine, Léo. Sur quelques propriétés des groupes d'automorphismes d'un groupe abstrait. C. R. Acad. Sci. Paris **230**, 2067–2069 (1950).

Let G be a group and H_1 and H_2 be two groups of automorphisms of G. As a lemma the author shows that for $a \in G$, $\alpha_1 \in H_1$, $\alpha_2 \in H_2$, any two of the following conditions imply the third: (1) every $\alpha_1(a)a^{-1}$ is fixed by H_2; (2) every $\alpha_2(a)a^{-1}$ is fixed by H_1; (3) $\alpha_1\alpha_2 = \alpha_2\alpha_1$. As an application of this lemma, it is shown that for a series $G = G_0 \supset G_1 \supset \cdots \supset G_{s+1} = 1$ of normal subgroups of G, the group A, of all automorphisms of G fixing every factor group G_i/G_{i+1}, is a nilpotent group of class at most 2^{s-1}. *M. Hall, Jr.* (Washington, D. C.).

Černikov, S. N. On the centralizer of a complete Abelian normal divisor in an infinite periodic group. Doklady Akad. Nauk SSSR (N.S.) **72**, 243–246 (1950). (Russian)

A periodic group is one with no infinite cyclic subgroup. Let every element in the periodic Abelian group A have infinite height (root-exponent). If A is a normal divisor of a periodic group P, and if every element of prime-square order in A is contained in the center of P, then P is Abelian. Let Q be an infinite periodic group possessing an ascending central series. If a single maximal Abelian normal divisor of Q is a special group, then Q is a special group. [The terms in the last sentence are defined in the author's article on special groups, Rec. Math. [Mat. Sbornik] N.S. **6**(**48**), 199–214 (1939); these Rev. **1**, 162.] *J. L. Brenner.*

Tschebotarjow, N. G. Das Resolventenproblem. Sowjetwissenschaft **1948**, no. 4, 140–152 (1948).

Translation of a paper in the Jubilee Symposium Devoted to the Thirtieth Anniversary . . . [Moscow-Leningrad, 1947; these Rev. **9**, 638].

Bruins, E. M. On transformation in quantum mechanics. Nederl. Akad. Wetensch., Proc. **53**, 440–445 = Indagationes Math. **12**, 92–97 (1950).

The author discusses some details of the known isomorphism between a group of collineations in a projective three space and the group of a quadric in a projective five-space. The former group has been characterised by Veblen and others in terms of linear families of correlations. The details given in this paper may be obtained in an invariant form, and more readily, by using the results of Veblen [Geometry of Complex Domains, Institute for Advanced Study Notes, Princeton, N. J., 1936]. *A. H. Taub* (Urbana, Ill.).

Garnir, H. Théorie de la représentation linéaire des groupes symétriques. Mém. Soc. Roy. Sci. Liège (4) **10**, 100 pp. (1950).

After brief preliminaries on group representation theory, Frobenius' determination of irreducible characters of symmetric groups is given in a more or less standard manner, first introducing auxiliary characters belonging to different diagrams, and then combining them to yield irreducible ones; orthogonality is used, as usual, to verify irreducibility. Then the recurrence theorem of Murnaghan [Amer. J. Math. **59**, 437–488 (1937)] and the reviewer [Jap. J. Math. **17**, 165–184 (1940); these Rev. **3**, 195] is given. Here, besides the reviewer's (right) hooks also Robinson's skew hooks [Amer. J. Math. **69**, 286–298 (1947); these Rev. **8**, 563] are employed. Schur's theorem of ramification is given as a special case of hooks of length 1. Then transposed diagrams and symmetric diagrams are considered by means of the ramification and recurrence theorems. To each tableau of Young is associated a polynomial which is the product of the alternates of variables corresponding to the indices in a same column. Theorems are given which describe the behaviour of the associated polynomials with respect to certain permutations of indices; these are to the effect that the associated polynomials of so-called standard tableaux form an integral basis for them, and also provide (a complete system of) irreducible representations with rational integral coefficients. The theorem of ramification is obtained from this construction. Then the author turns to analyzing the ramifications of a standard tableau and describing transpositions of pairs of indices in terms of them; this gives a proof of Young's theorem on the construction of irreducible orthogonal representations of a symmetric group. The paper is augmented with many illustrative examples, including the construction of irreducible orthogonal representations of S_4 and S_5. *T. Nakayama* (Urbana, Ill.).

Kuranishi, Masatake. On Euclidean local groups satisfying certain conditions. Proc. Amer. Math. Soc. **1**, 372–380 (1950).

Let G be a Euclidean local group. The author shows that if G has a neighborhood U of 1 in which a metric $\rho(x, y)$ can be introduced such that (A) $K_2\rho(y, 1) \leqq (xy, x) \leqq K_1\rho(y, 1)$ when $x, y, xy \in U$ and (B) $K_4 2^n \rho(x, y) \leqq \rho(x^{2^n}, y^{2^n}) \leqq K_3 2^n \rho(x, y)$ when $x, x^2, x^4, \cdots, x^{2^n} \in U$ and $y, y^2, \cdots, y^{2^n} \in U$, $K_1, \cdots, K_4$ being positive constants, then G is a local Lie group and conversely. The conditions (A), (B) are used to select a uniformly converging subsequence from $P_n(x, y) = (x^{1/k} y^{1/k})^k$, $k = 2^n$. The limit function $P(x, y)$ is shown to be the composition function of an Abelian group which is isomorphic to an r-dimensional vector group H where $r = \dim G$. Inner automorphisms of G correspond to linear automorphisms of H and it follows that G is a local Lie group if its center is discrete. The author disposes of the case with nondiscrete center N by showing the existence of a coordinate system in which the composition function of G is of class C^3 (using, of course, the fact that N and G/N are local Lie groups). *P. A. Smith* (New York, N. Y.).

Kawada, Yukiyosi. On the invariant differential forms of local Lie groups. J. Math. Soc. Japan **1**, 217–225 (1949).

Let G be a local Lie group. Denote by $I(G)$ and $J(G)$ the rings of left invariant and of two-sided invariant differential-forms on G. Let $p(k)$ be the dimension of the space of homogeneous elements of degree k of $J(G)$; then $P(t) = \sum_k t^{p(k)}$ is the Poincaré polynomial of G. If G is compact, then it follows from the theorem of Hopf that $P(t) = \prod_{i=1}^{l}(1 + t^{m(i)})$, the exponents $m(i)$ being odd integers. This result is extended to all local groups G which satisfy the following condition: the representation of G given by $I(G)$ is semi-simple. As in Hopf's method, the proof is carried out by means of the homomorphism of $I(G)$ into $I(G \times G)$ which corresponds to the mapping $(x, y) \to xy^{-1}$ of $G \times G$ into G. [Note by the reviewer. The same result has been proved in a purely algebraic manner (using only the Lie algebra and not the local group) by Koszul [Bull. Soc. Math. France **78**, 65–127 (1950); these Rev. **12**, 120]. The two papers are obviously independent of each other.] *C. Chevalley.*

Grabar', L. P. A theorem on bicompact groups. Mat. Sbornik N.S. **27**(**69**), 139–142 (1950). (Russian)

Let G be a bicompact group. The author shows that every neighborhood U of the identity of G contains a normal subgroup N of G such that G/N satisfies the second axiom of countability. This theorem is a well-known consequence of the existence of sufficiently many representations of G. The

author's proof is direct and elementary, however, and makes no use of the theory of representations or of integration on the group. *G. W. Mackey* (Cambridge, Mass.).

Zippin, Leo. Two-ended topological groups. Proc. Amer. Math. Soc. **1**, 309–315 (1950).

Let G be a locally compact connected topological group. An end of a space (at least under some restrictions) may be described by saying that an end is determined by a diverging sequence of points x_n, where x_n and y_n determine the same end if x_n and y_n are in a connected set K_n with K_n diverging. It is known that a group G as above has at most two ends. A group which has zero ends is compact. The main result of the present paper is the following. If a locally compact, connected topological group G is two-ended, then G contains a closed subgroup T isomorphic to the group of reals such that the coset-space G/T is compact; moreover, the space G is the topological product of the axis of reals by a compact connected set homeomorphic to the space G/T. Let the two ends be e_L and e_R. The proof uses the set C of those elements of G contained in compact subgroups, the set R of elements x such that x^n approaches e_R, and a corresponding set L. The principal lemma states that if g_n approaches e_R then for almost all n, g_n is in R. *D. Montgomery.*

Serre, Jean-Pierre. Compacité locale des espaces fibrés. C. R. Acad. Sci. Paris **229**, 1295–1297 (1949).

Let the topological group G act on the topological space E. That is, for every $s \in G$ let there be given a homeomorphism $x \to s(x)$ of E into itself such that $s(x)$ is continuous in the two variables s and x together and $(st)(x) = s(t(x))$. The orbits in E constitute a fibering of E; it is supposed that each orbit is closed. The fibering is called a principal fibering if whenever x and x' belong to the same orbit there exists only one element $s = u(x, x')$ of G such that $x' = s(x)$ and the mapping $\langle x, x' \rangle \to u(x, x')$ is continuous. The author proves in the case of a principal fibering that if the group G and the quotient space of fibers are both locally compact then E is locally compact. A corollary is the theorem of Vilenkin [Mat. Sbornik N.S. **22(64)**, 135–177 (1948); these Rev. **9**, 497] that every extension of a locally compact group by a locally compact group is itself locally compact.

L. H. Loomis (Cambridge, Mass.).

Serre, Jean-Pierre. Trivialité des espaces fibrés. Applications. C. R. Acad. Sci. Paris **230**, 916–918 (1950).

The author shows that in certain cases a principal fibering [see the preceding review] admits a cross-section. This occurs, in particular, when G is a projective limit of a countable number of Lie groups and the base (space of fibers) is locally compact, paracompact, and contractible to one of its points. Various topological consequences are drawn. For example, if G is the projective limit of a countable number of Lie groups and is also simply-connected, and if g is a closed arc-wise connected normal subgroup, then g and G/g are both simply-connected. *L. H. Loomis.*

Borel, Armand. Sections locales de certains espaces fibrés. C. R. Acad. Sci. Paris **230**, 1246–1248 (1950).

The author shows that the theorems of Serre [see the preceding review] remain valid if the system of Lie groups of which G is the projective limit is no longer assumed to be countable.

L. H. Loomis (Cambridge, Mass.).

Graev, M. I. Theory of topological groups. I. Norms and metrics on groups. Complete groups. Free topological groups. Uspehi Matem. Nauk (N.S.) **5**, no. 2(36), 3–56 (1950). (Russian)

This is an expository essay uniting a variety of topics on the structure of topological groups. The principal topics are named in the title in the order of the three parts into which the paper is divided. Almost all of the work belongs to the last decade. *L. Zippin* (Flushing, N. Y.).

Vilenkin, N. Ya. The theory of topological groups. II. Direct products. Direct sums of groups of order 1. Locally compact Abelian groups. Fibered and weakly separable groups. Uspehi Matem. Nauk (N.S.) **5**, no. 4(38), 19–74 (1950). (Russian)

Nominally the continuation of a two-part report on topological groups [see the preceding review], this part is quite independent of the first paper. Like the first it is an expository essay on topics named in the title, containing very many details and a number of full proofs. Almost all of the subject matter of this paper was developed in the last five years, much of it by the author or by the author and others, independently.

An introduction reviews the brief history, and sketches the contents of the paper. Part I is concerned with the idea of a direct product of an arbitrary number of topological groups, and is devoted to definitions and general properties. One of the principal applications in this section is to the duality-theory associated with character groups (and here recent work of S. Kaplan [Duke Math. J. **15**, 649–658 (1948); these Rev. **10**, 233] is followed in part). Part II analyzes the structure of certain classes of groups which are direct sums (the first part of the paper, for the sake of generality, frequently permits groups to be non-Abelian) of groups of rank 1; there are a variety of cases according as one or more of the four types of groups of rank 1 are admitted as factors. Part III is on the algebraic theory of locally bicompact Abelian groups, with principal emphasis on the null-dimensional groups. Here are studied the universal groups for various classes of groups, the types of conditions for various classes of groups under which a subgroup can be recognized as a direct summand, the analogues of the Ulm theory for countable periodic groups, the notion of structural isomorphisms, work of Grayev [Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 125–144 (1947); these Rev. **8**, 500] generalizing work of Baer [Amer. J. Math. **61**, 1–44 (1939)]. Part IV is on the class of groups associated principally with the author's name: the separable, co-separable, and weakly-separable, and, most recently, the fibred group. The most recent of the papers underlying this last section, and references to earlier work, including the material of the first parts of this paper, will be found in two earlier papers of the author [Mat. Sbornik N.S. **22(64)**, 135–177 (1948); **24(66)**, 189–226 (1949); these Rev. **9**, 497; **10**, 679] as well as in the paper reviewed below. *L. Zippin.*

Vilenkin, N. Ya. On the classification of separable and coseparable topological Abelian groups. Mat. Sbornik N.S. **27(69)**, 85–102 (1950). (Russian)

The author defines an extensive class of groups which admit a theory completely analogous to that of the countable, primary, periodic groups. These are the normal, coseparable primary groups to be discussed. Another, similarly directed class of groups was announced previously by the author [Doklady Akad. Nauk SSSR (N.S.) **61**, 969–971

(1948); these Rev. **10**, 282]. Let G denote a coseparable primary group associated with a fixed prime p (notation and definitions as in [Rec. Math. [Mat. Sbornik] **19**(**61**), 85–154, 311–340 (1946); Mat. Sbornik N.S. **22**(**64**), 135–177, 191–192 (1948); these Rev. **8**, 132, 312; **9**, 497]. Consider the natural descending chain $G=G_0\supset G_1\supset\cdots\supset G_\alpha\supset\cdots\supset G_\tau$, τ necessarily of some countable type, where $G_\alpha=p^\alpha G_{\alpha-1}$ or $=\prod G_\beta$, $\beta<\alpha$, according as α has or has not a predecessor. Let $p^\alpha H$ denote the set of elements of H of the form $p^\alpha y$, y varying over H. Let $p^\alpha H$ denote the intersection of closures of the $p^\alpha H$. Let ${}_\alpha[H]$ denote the elements of H of order p^α and let ${}_\infty[H]$ denote the elements of infinite order. A subgroup H of G is called "just-right" (vpolnye pravil'no) if for every transfinite α and every integer n

$$p^{\alpha+n}G\cap p^nH=p^n(p^\alpha G\cap H).$$

Here, if α is of the second kind, $\alpha=\omega\gamma$, then $p^\alpha G=G_\alpha$, and if $\alpha=\omega\gamma+n$, then $p^\alpha G=p^nG_\alpha$. The group G is called normal if G has a subgroup H such that (1) H is open, (2) H is just-right, (3) ${}_\infty[H]=0$. The need for some restriction on the class of coseparable groups, for the point of view of this work, is shown by the author's example of two coseparable, nonisomorphic groups G and G' such that the factor groups $G_\alpha/G_{\alpha+1}$ and $G'_\alpha/G'_{\alpha+1}$ are isomorphic for every $\alpha\leqq\tau=\tau'$.

L. Zippin (Flushing, N. Y.).

Vilenkin, N. Ya. Direct spectra of topological Abelian groups and their limit groups. Doklady Akad. Nauk SSSR (N.S.) **72**, 617–620 (1950). (Russian)

The author defines a concept of bounded structure in (Abelian) groups which in the case of topological spaces he credits to Sze-Tsen Hu [J. Math. Pures Appl. (9) **28**, 287–320 (1949), p. 288; these Rev. **11**, 482]. The group G has a set $T=(M_\alpha)$ of subsets M_α which are called bounded and are such that: (1) if A is bounded and $B\subset A$, then B is bounded; (2) if A and B are bounded so are the sets $A\cup B$ and AB [the reviewer takes this to be the set of products, since $A\cap B$ is used elsewhere in the paper for the common part]; (3) if A is bounded, then A^{-1} is bounded; (4) every point is bounded. Given two groups G and G_1, each with bounded structure, and a homomorphism f of G into G_1, the author calls f bounded if it takes bounded sets into bounded ones, calls f boundable if every bounded set in the image is in the image of a bounded set, and finally doubly-bounded if f has both properties.

If X denotes the group of continuous characters of G, and $A\subset G$, then $N(A)\subset X$ is the set of characters x for which $|x(g)|<\frac{1}{4}$ for every $g\in A$. These $N(A)$, for bounded A in G, are used to define the topology of the character-group X. If U is a neighborhood of e (identity) in G, $N(U)$ is taken as a bound in X. A set $\bar{A}$ of G is called a quasi-convex (q.-c.) envelope of $A\subset G$ if it consists of all elements a such that $|x(A)|<\frac{1}{4}$ implies $|x(a)|<\frac{1}{4}$ for every $x\in X$; a set is q.-c. if it is its own envelope. The group G is locally q.-c. if it has a complete system of q.-c. neighborhoods of e; G has q.-c. bounded structure if every bounded set belongs to a bounded set which is also q.-c. A lemma asserts that for $A\subset G$ sets $N(A)$ are q.-c. in X; consequently, locally-bicompact groups (Abelian) are locally q.-c.

Given G and X, let G^* be the group of characters of X and let α denote the mapping: $g\to g^*$ determined by the relation: $x(g)=g^*(x)$ for all $x\in X$; α is an algebraic homomorphism. Theorem 1 states that α is continuous and bounded and that if G is locally q.-c. with q.-c. bounded structure then α is open and boundable. If α is a full isomorphism of G onto G^*, G is called involutory. Theorem 2 says that in order that a closed subgroup H of G be involutory it is necessary and sufficient that (1) every character of H may be extended to a character of G and (2) that H be q.-c. in G. A corollary to Theorem 3 asserts that the limit group of an inverse spectrum of locally bicompact groups is an involutory group if the bounded sets giving rise to the topologies of the character groups are taken as the closures of the bicompact sets.

A second half of this abstract is devoted to direct spectra and to a theorem of duality. The essential definitions are these. Let G be a limit of the direct spectrum $\{G_\lambda; T_\mu^\lambda\}$ of topological groups G_λ. The elements of G are the collections (x_λ), almost all $x_\lambda=e$. Let U_λ be q.-c. neighborhoods of e in G such that for $\lambda<\mu$, $U_\lambda=T_\mu^{\lambda-1}[U_\mu\cap T_\mu^\lambda(G_\lambda)]$. Let $U^0(U_\lambda)$ denote the totality of elements of G with $x_\lambda\in U_\lambda$ and such that $\sum_\lambda x_\lambda/U_\lambda<1$; here the notation follows Kaplan [Duke Math. J. **15**, 649–658 (1948), p. 650; these Rev. **10**, 233]. The set $U^0(U_\lambda)$ is called a kernel of G. The set A is called open if to each a of A there can be found a kernel U such that $a+U\subset A$; this gives a topology to G. Let H be the subgroup generated by elements of the form $x_\lambda-T_\mu^\lambda x_\lambda$. The factor-group $\hat{G}=G/\bar{H}$ is called the topological limit group of the direct spectrum. The T_2-separation axiom holds in $\hat{G}$ but not necessarily in G. If the G_λ are locally q.-c., if the homomorphisms T_μ^λ are bounded, and if the character groups are topologized through the bounded sets, then the limit group of the inverse spectrum of character groups is fully isomorphic to the character group of $\hat{G}$ (theorem 7).

L. Zippin (Flushing, N. Y.).

NUMBER THEORY

van Wijngaarden, A. A table of partitions into two squares with an application to rational triangles. Nederl. Akad. Wetensch., Proc. **53**, 869–881 = Indagationes Math. **12**, 313–325 (1950).

This paper furnishes a table giving integral solutions (x, y), $0\leqq x\leqq y$, of (1) $n=x^2+y^2$ for each integer $n\leqq 10\,000$ for which (1) is possible. The table is used to solve the problem of finding the smallest triangle having integral sides and rational medians. The answer to the problem is Euler's example (68, 85, 87) with medians (79, 131/2, 127/2).

D. H. Lehmer (Berkeley, Calif.).

Pizá, Pedro A. The powers of triangular numbers. Revista Unión Mat. Argentina **14**, 256–258 (1950). (Spanish)

The author expresses the nth power of the triangular number $\frac{1}{2}x(x+1)$ as follows:

$$[\tfrac{1}{2}x(x+1)]^n=(1/2^{n-1})\sum\binom{n}{2i-1}S_x^{2n+1-i},$$

where $S_x^m=\sum_{a=1}^{x}a^m$ and where the summation is over all i for which the binomial coefficient has a meaning.

H. W. Brinkmann (Swarthmore, Pa.).

Andree, Richard V. The number of solutions of the totient equation $\varphi(x)=B$. Pi Mu Epsilon J. 1, 47–48 (1950).

Aigner, Alexander. A generalization of Gauss' lemma. Amer. Math. Monthly 57, 408–410 (1950).

A number is said to belong to the first half mod n if it is congruent mod n to one of the numbers $1, 2, \cdots, \frac{1}{2}(n-1)$, and to the second half if it is congruent mod n to one of the numbers $[\frac{1}{2}n]+1, [\frac{1}{2}n]+2, \cdots, n-1$. The author shows that if p is an odd prime and A is odd then the number of terms in the sequence $A, 2A, \cdots, \frac{1}{2}(p-1)A$, which belong to the second half mod p is equal to the number of terms which belong to the second half mod $2p$. By means of this result he gives a simple variant of Gauss' third proof of the quadratic reciprocity theorem. *R. Bellman.*

Nicolesco, G. Sur quelques équations diophantiennes. Mathesis 59, 95–98 (1950).

Identities are given which solve certain Diophantine equations, such as the sum of n squares equal to a cube. The question of completeness of solutions is not raised. *I. Niven* (Eugene, Ore.).

Nagell, Trygve. Über die Anzahl der Lösungen gewisser diophantischer Gleichungen dritten Grades. Math. Z. 52, 750–757 (1950).

Consider the equation $ax^3+bx^2y+cxy^2+dy^3=k$, the cubic form having negative discriminant. It is proved that this equation has at most 6 integral solutions when $k=2$, at most 12 when $k=4$. This result is improved in many special cases, of which only one is reproduced here. If $k=2$ or 4 there are at most 2 solutions in case $a\equiv d\equiv b+1\equiv c+1\equiv 1 \pmod 2$. The method used is reduction of cubic forms to simpler equivalent ones. Use is also made of the author's earlier results of a similar kind for $k=1$ [Math. Z. 28, 10–29 (1926)]. *I. Niven* (Eugene, Ore.).

Bellman, Richard. On some divisor sums associated with Diophantine equations. Quart. J. Math., Oxford Ser. (2) 1, 136–146 (1950).

Let $d(n)$ and $d_j(n)$ be respectively the number of divisors of n and the number of representations of n as a product of j integers. The following asymptotic formulas are obtained:

$$S_3=\sum_{n\leq N} d(n)d_3(n+r)\sim c_3(r)N\log^3 N$$

and

$$S_4=\sum_{n\leq N} d(n)d_4(n+r)=c_4(r)N\log^4 N.$$

The first step is the reduction of the problem to that of estimating the sums $T_j=\sum_{n\leq N}d_j(kn+r)$ for $j=3, 4$. Then an application of the Perron formula gives rise to a Dirichlet series, and the calculation of the mean value of the fourth power of this series constitutes the main part of the proof. Details are given for the case $r=1$, with an indication at the end of the paper of the necessary modifications for the general case. A heuristic treatment of S_3 was given by Titchmarsh [same J., Oxford Ser. (1) 13, 129–152 (1942); these Rev. 4, 131] following the circle method of Hardy and Littlewood. *I. Niven* (Eugene, Ore.).

Gloden, A. Resolution de la congruence $X^4+1\equiv 0 \pmod{p^2}$ avec une table. Euclides, Madrid 10, 74 (1950).

The author shows how to solve the congruence (1) $X^4+1\equiv 0 \pmod{p^2}$ given a solution with the modulus p. The process involves solving a quadratic congruence modulo p^2. Two solutions are given of the congruence (1) for all possible primes less than 200, eight primes in all. *D. H. Lehmer* (Berkeley, Calif.).

Halberstam, H. Representation of integers as sums of a square, a positive cube, and a fourth power of a prime. J. London Math. Soc. 25, 158–168 (1950).

This paper sharpens a result of K. F. Roth [J. London Math. Soc. 24, 4–13 (1949); these Rev. 10, 431]. It is proved that almost all positive integers n are representable in the form $n=x_1^2+x_2^3+p^4$, where x_1 and x_2 are positive integers and p is a prime. The method employs the technique of Farey dissection and follows the lines of Roth's analysis. Among the new tools required are some results of Hua [Quart. J. Math., Oxford Ser. (1) 9, 68–80 (1938)] in additive prime number theory. *A. L. Whiteman.*

Ostmann, Hans-Heinrich. Über die Anzahl der Elemente von Summenmengen. J. Reine Angew. Math. 187, 222–230 (1950).

Let A, B be two sets of nonnegative integers with densities α and β. Let C be their sum-sequence with density γ. Then a well-known theorem [H. B. Mann, Ann. of Math. (2) 43, 523–527 (1942); these Rev. 4, 35] states that $\gamma\geqq\min[1, \alpha+\beta]$. The main result of the present paper is that there exist pairs of sequences for which $\gamma=\alpha+\beta$, provided at least one of α and β is irrational. The proof is elementary but somewhat complicated. Another new result is a sharpened form of a theorem of Scherk [Ann. of Math. (2) 42, 538–546 (1941); these Rev. 2, 349] which is too long to be stated here. *F. J. Dyson* (Princeton, N. J.).

Erdös, P. On almost primes. Amer. Math. Monthly 57, 404–407 (1950).

An "almost prime" is defined to be an odd composite number n such that $2^{n-1}\equiv 1 \pmod n$. The number $f(x)$ of almost primes up to x is proved to be less than $x\exp(-\frac{1}{3}(\log x)^{\frac{1}{3}})$, and so is much less than the number of primes up to x. The proof is by a division into four classes. The author conjectures that $f(x)$ is really of much smaller order. *H. Davenport* (London).

Shapiro, Harold N. On the number of primes less than or equal x. Proc. Amer. Math. Soc. 1, 346–348 (1950).

From $n!=\prod_{p\leq n}p^{\alpha_p}$ where $\alpha_p=[n/p]+[n/p^2]+\cdots$ and Stirling's formula in the form $\log n!=n\log n+O(n)$ it follows that

$$\sum_{p\leq x}\left[\frac{x}{p}\right]\log p=x\log x+O(x). \tag{1}$$

From this result Landau [Handbuch der Lehre von der Verteilung der Primzahlen, Teubner, Leipzig-Berlin, 1909, § 18] deduced Čebyšev's inequality $\vartheta(x)\leqq\beta x$; to obtain $\vartheta(x)\geqq\alpha x$ he had to use the sharper form of (1) with $x\log x-x+o(x)$ on the right-hand side. The author deduces both Čebyšev inequalities from (1) alone as a particular case of the following theorem which he proves by elementary methods. Let $a_n\geqq 0$ be a sequence of real numbers such that

$$\sum_{n\leq x}\left[\frac{x}{n}\right]a_n=x\log x+O(x).$$

Then for $S(x)=\sum_{n\leq x}a_n$ there exist two constants α, β such that for all sufficiently large x, $\alpha x\leqq S(x)\leqq\beta x$. *R. A. Rankin* (Cambridge, England).

Shapiro, Harold N. On primes in arithmetic progressions. I. Ann. of Math. (2) **52**, 217–230 (1950).

The author establishes by elementary methods the analogue of Selberg's formula for arithmetical progressions, namely,

$$\text{(A)}\quad \sum_{\substack{p\le x\\ p\equiv B\,(A)}} \log^2 p+\sum_{\substack{pq\le x\\ pq\equiv B\,(A)}} \log p\log q=\frac{2}{\varphi(A)}x\log x+O(x),$$

where $(B, A)=1$ and p and q are primes. This formula together with

$$\text{(B)}\quad \sum_{\substack{p\le x\\ p\equiv 1\,(A)}} \frac{\log p}{p}\to\infty,$$

which is obtained by using certain results of algebraic number theory applied to the field obtained from the rational field by adjoining the Ath roots of unity, is used to deduce that

$$\sum_{\substack{p\le x\\ p\equiv B\,(A)}} \log^2 p\to\infty$$

[in (1.4) $\log p$ should be replaced by $\log^2 p$], from which Dirichlet's theorem follows. In addition, it can be shown from (A) without the use of (B) that at least half of the progressions modulo A (which are relatively prime to A) contain an infinity of primes. In the special case when $A=p^\alpha$ $(p\ge 2)$ it is possible to deduce from this and (A) that, if B is a quadratic nonresidue modulo A, then there are infinitely many primes congruent to B (mod A).

R. A. Rankin (Cambridge, England).

Shapiro, Harold N. On primes in arithmetic progression. II. Ann. of Math. (2) **52**, 231–243 (1950).

The author considers the series

$$L_0(\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n},\quad L_1(\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)\log n}{n},$$

where χ is a nonprincipal character modulo A. It can be shown by elementary methods that, if χ is real, $L_0(\chi)\neq 0$. A sketch of the proof given by Landau [Handbuch der Lehre von der Verteilung der Primzahlen, Teubner, Leipzig-Berlin, 1909, § 106] is given. By means of repeated applications of a generalisation of the Möbius inversion formula and by considering $L_0(\chi)$ and $L_1(\chi)$ it is shown that

$$\text{(C)}\quad \sum_{p\le x}\frac{\chi(p)\log p}{p}=\begin{cases} O(1) \text{ if } L_0(\chi)\neq 0,\\ -\log x+O(1) \text{ if } L_0(\chi)=0.\end{cases}$$

From this it is deduced that

$$\varphi(A)\sum_{\substack{p\le x\\ p\equiv 1\,(A)}}\frac{\log p}{p}=(1-N)\log x+O(x),$$

where N is the number of nonprincipal characters χ for which $L_0(\chi)=0$. Since the left-hand side is positive, $N=0$ or 1. This means that $L_0(\chi)\neq 0$ for a complex character, as otherwise it would follow that $N\ge 2$ since $L_0(\bar\chi)=\overline{L_0(\chi)}$ and χ and $\bar\chi$ are different characters. (C) then gives

$$\sum_{p\le x}p^{-1}\chi(p)\log p=O(1),$$

and it follows easily that

$$\text{(D)}\quad \sum_{\substack{p\le x\\ p\equiv B\,(A)}}\frac{\log p}{p}=[\varphi(A)]^{-1}\log x+O(1).$$

The only processes in the deduction of (D) which are not strictly elementary are those involving the use of the infinite series $L_0(\chi)$ and $L_1(\chi)$. A sketch of a method of proof in which these are replaced by finite sums is given. Let

$$\vartheta_B(x)=\sum_{\substack{p\le x\\ p\equiv B\,(A)}}\log p.$$

By means of (D) and (A) [see the preceding review] the author shows that

$$\liminf \vartheta_B(x)/x+\limsup \vartheta_B(x)/x=2/\varphi(A)$$

for each B with $(B, A)=1$, and that the lower limit, and therefore the upper limit, is the same for each B. From the prime ideal theorem (this can be proved by elementary methods; see the author [Comm. Pure Appl. Math. **2**, 309–323 (1949); these Rev. **11**, 501]) it can be deduced that, for $B=1$, each limit is $1/\varphi(A)$ and from this it follows that $\vartheta_B(x)\sim x/\varphi(A)$ for all $(B, A)=1$. *R. A. Rankin.*

Rédei, L. Kurzer Beweis eines Satzes von Vandiver über endliche Körper. Publ. Math. Debrecen **1**, 99–100 (1949).

The theorem in question is as follows. Let $q=1+mc$ and let g be a generator of the multiplicative group of $GF(q)$. Let (i, j) denote the number of solutions of $1+g^{i+rm}=g^{j+sm}$ $(r, s=0, \cdots, c-1)$. Let $A_{hk}=\sum_{i,j=0}^{m-1}(i, j)(i+h, j+k)$. Then

$$A_{hk}=\begin{cases}(c-1)^2+c(m-1), & m\mid h, k\\ c^2, & m\nmid h, k, h-k,\\ c^2-c, & \text{otherwise.}\end{cases}$$

L. Carlitz (Durham, N. C.).

Faircloth, O. B., and Vandiver, H. S. On multiplicative properties of a generalized Jacobi-Cauchy cyclotomic sum. Proc. Nat. Acad. Sci. U. S. A. **36**, 260–267 (1950).

In a previous paper, one of the authors has introduced a generalized Jacobi sum ψ as follows. Let $F(p^n)$ be a finite field, let $m_1, \cdots, m_s$ be factors of p^n-1, and define $\operatorname{ind} a$ for $a\neq 0$ by $g^{\operatorname{ind} a}=a$, g being a primitive root of $F(p^n)$; then set

$$\psi=\psi(\alpha_1^{a_1}, \cdots, \alpha_s^{a_s})=\sum\alpha_s^{a_s\operatorname{ind} A}\prod_{i=1}^{s-1}\alpha_i^{a_i\operatorname{ind} a_i},$$

where $\alpha_i=\exp(2\pi i/m_i)$ and the summation is over all a_i in $F(p_n)$ with $A=1-\sum_{i=1}^{i-1}a_i$ [Vandiver, same vol., 144–151 (1950); these Rev. **11**, 330]. In the present paper certain multiplicative properties of ψ are derived. These turn out to be unexpectedly simple and are in the nature of factorizations of the algebraic integers involved. A simple example is the following. Set $\psi_{i,s}=\psi(\prod_{i=1}^{i}\alpha_i^{a_i}, \alpha_{i+1}^{a_{i+1}}, \cdots, \alpha_s^{a_s})$; then in case the numbers m_i involved in the definition just given are relatively prime in pairs, we have $\psi_{1,s}=\psi_{1,2}\psi_{2,3}\cdots\psi_{s-1,s}$.

H. W. Brinkmann (Swarthmore, Pa.).

Kelly, John B. A closed set of algebraic integers. Amer. J. Math. **72**, 565–572 (1950).

Let θ be an algebraic integer whose conjugates (except θ itself) have all moduli less than one. It is known that the set S_1 of all numbers θ is closed. The object of the paper is the proof of the following theorem. Let ξ be an imaginary algebraic integer whose conjugates (except ξ and $\bar\xi$) have all moduli less than one. Then the limit points of the set S_2 of all numbers ξ belong to S_2 if the limit points are imaginary, and to S_1 if the limit points are real. Thus, S_2+S_1 is closed.

R. Salem (Cambridge, Mass.).

Lochs, Gustav. Über die Lösungszahl einer linearen, diophantischen Ungleichung. Jber. Deutsch. Math. Verein. **54**, 41–51 (1950).

Let $\alpha_1, \cdots, \alpha_k$ be linearly independent real numbers, and let the numbers $\sum_{r=1}^{k} x_r\alpha_r$ ($x_1, \cdots, x_k$ nonnegative integers), arranged in increasing order of magnitude, be denoted by $\lambda_1, \lambda_2, \cdots$. It is shown that the number $n(\alpha_1, \cdots, \alpha_k; x)$ of solutions of the inequality $\lambda_n \leqq x$ is

$$(1)\quad (k!\alpha_1 \cdots \alpha_k)^{-1}(x^k + \tfrac{1}{2}k(\alpha_1 + \cdots + \alpha_k)x^{k-1} + o(x^{k-2})),$$

which implies that

$$(2)\quad \lambda_n = (k!\alpha_1 \cdots \alpha_k \cdot n)^{1/k} - \tfrac{1}{2}(\alpha_1 + \cdots + \alpha_k) + o(1).$$

It is also shown that the term $o(x^{k-2})$ of (1) cannot be replaced by $o(x^{k-1}\zeta(x))$ if $\zeta(x) = o(1)$. A consequence of (2), obtained by taking $\alpha_i = \log p_i$, is that the nth positive integer (ordered by size) which is composed exclusively of the primes $p_1, \cdots, p_k$ is asymptotically equal to

$$(k! \log p_1 \cdots \log p_k \cdot n)^{1/k}/(p_1 \cdots p_k)^{\frac{1}{k}}.$$

The method is similar to that used by D. H. Lehmer [Duke Math. J. **7**, 341–353 (1940); these Rev. **2**, 149] to obtain bounds for $n(\alpha_1, \cdots, \alpha_k; x)$. *W. J. LeVeque.*

Blaney, Hugh. Some asymmetric inequalities. Proc. Cambridge Philos. Soc. **46**, 359–376 (1950).

Let $\xi = \alpha x + \beta y$, $\eta = \gamma x + \delta y$ be two real linear forms of determinant Δ, and μ a number with $-1 < \mu \leqq 1$. The object of this article is to determine real continuous functions $g(\mu)$ for which the inequalities, $-\mu g(\mu)\Delta < (\xi+\theta)(\eta+\phi) \leqq g(\mu)\Delta$, have integral solutions x, y for all real θ and ϕ and to determine the values of μ for which these inequalities are the best possible (i.e., values of μ such that these inequalities are no longer true in the strict sense for any integral x, y for some ξ, η, θ, ϕ). For $0 < \mu \leqq 1$ the author shows that the inequalities have solutions if $g(\mu) = \frac{1}{4}\mu^{-1}$ and that they are best possible for $\mu = m/(m+2)$, $m = 1, 2, 3, \cdots$. Similar results are obtained if $g(\mu) = [(1+\mu)(1+9\mu)]^{-\frac{1}{2}}$, $0 \leqq \mu \leqq \frac{1}{3}$, in which case the inequalities are best possible for $\mu = 1/(4m-1)$, $m = 1, 2, 3, \cdots$, and $\mu = 0$. For negative μ a function $g(\mu)$ is given for which the inequalities are best possible for only a finite number of values μ. Using geometric methods, Davenport [Acta Math. **80**, 65–95 (1948); these Rev. **10**, 101] determined the first of these three functions $g(\mu)$ without enumerating all the values of μ for which the inequalities are the best possible. The author makes use of Davenport's method in this case and an adaptation of it to develop the third $g(\mu)$. All his other proofs are arithmetic.

D. Derry (Vancouver, B. C.).

Sawyer, D. B. A note on the product of two non-homogeneous linear forms. J. London Math. Soc. **25**, 239–240 (1950).

The theorem of Davenport [Acta Math. **80**, 65–95 (1948); these Rev. **10**, 101] that integer values ξ, η exist such that $-\lambda\Delta/4 \leqq xy < \Delta/4\lambda$, where $x = a\xi + b\eta + p$, $y = c\xi + d\eta + q$ with a, b, c, d, p, q real and $\Delta = ad - bc > 0$, is demonstrated by a slight modification of a proof given by the author when $\lambda = 1$ [J. London Math. Soc. **23**, 250–251 (1948); these Rev. **10**, 355]. This is generalized to a set of four inequalities of the form $|x^{\theta}y^{1-\theta}| < \lambda\Delta^{\frac{1}{2}}$ (with equality to be allowed in one of them), one for each quadrant, one of which is satisfied by a point (x, y). Here θ and λ depend on the quadrant, $0 < \theta < 1$, and the θ's and λ's satisfy an inequality. *L. Tornheim.*

Delone, B. N. Algorithmus der zerteilten Parallelogramme. Sowjetwissenschaft **1948**, no. 2, 178–210 (1948).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Mat. **11**, 505–538 (1947); these Rev. **9**, 334.

Prasad, A. V. Simultaneous Diophantine approximation. Proc. Indian Acad. Sci., Sect. A. **31**, 1–15 (1950).

Let $K_n^{(X)}$ be the region in $(n+1)$-dimensional space defined by $n^{-1}\sum_{r=1}^{n} x_r^2 |x_0|^{2/n} \leqq 1$, $n^{-1}\sum_{r=1}^{n} x_r^2 \leqq X^2$. It is shown for certain numbers C_n'' with $C_n'' \sim n^{\frac{1}{2}}/(\pi e)^{\frac{1}{2}n}$ 3.763 ... that, provided X is sufficiently large, every lattice with determinant less than $(C_n'')^{-1}$ has a point other than the origin in $K_n^{(X)}$. In this result the constant 3.763 ... is larger than the constant $2e^{\frac{1}{2}} = 3.297$ ... which can be obtained by applying a result of Blichfeldt [Trans. Amer. Math. Soc. **15**, 227–235 (1914), theorem II] to an $(n+1)$-dimensional sphere inscribed in $K_n^{(X)}$. By simple applications of his result the author obtains a theorem on the simultaneous Diophantine approximations to n irrational numbers and a theorem on the approximate solution in integers of a homogeneous linear equation in $n+1$ variables.

C. A. Rogers (London).

Raisbeck, Gordon. Simultaneous Diophantine approximation. Canadian J. Math. **2**, 283–288 (1950).

Given any set of real numbers $z_1, z_2, \cdots, z_n$ and an integer t we can find an integer q, $1 \leqq q \leqq t^{n-1}(t-1) - 2^{n-1} + 1$, and a set of integers $p_1, p_2, \cdots, p_n$ such that $|qz_j - p_j| \leqq 1/t$, $j = 1, 2, \cdots, n$. This result is sharper than the classical results of Dirichlet-Minkowski and than a theorem of Pipping [Acta Acad. Aboensis **13**, no. 9 (1942); these Rev. **7**, 274]. For $n = 2$ the author proves that his result is the best possible. *J. F. Koksma* (Amsterdam).

Hartman, S. Une généralisation d'un théorème de M. Ostrowski sur la répartition des nombres mod 1. Ann. Soc. Polon. Math. **22** (1949), 169–172 (1950).

L'auteur démontre la généralisation suivante d'un théorème d'Ostrowski [Jber. Deutsch. Math. Verein **39**, 34–46 (1930)]. Soit I un intervalle de longueur $\leqq 1$, fermé à gauche et ouvert à droite et réduit mod 1, c'est-à-dire que tout point z de I a été remplacé par $z - [z]$. Etant donnés n nombres réels $\alpha_1, \cdots, \alpha_n$ et n tels intervalles $I_1, \cdots, I_n$, le nombre $N(\alpha_1, \cdots, \alpha_n, x, I_1, \cdots, I_n)$ pour tout entier $x \geqq 1$ désignera le nombre des nombres entiers y, $1 \leqq y \leqq x$, pour lesquels simultanément $\alpha_i y - [\alpha_i y] \in I_i$ ($i = 1, 2, \cdots, n$). Théorème. (A) Étant donnés deux systèmes de nombres réels $\alpha_1, \cdots, \alpha_n$ et $\xi_1, \cdots, \xi_n$ ($0 < \xi_i \leqq 1$), où l'un au moins des ξ_i est irrationnel, et un nombre naturel x, alors: (I) Il existe deux systèmes d'intervalles $I_1', \cdots, I_n'$ et $I_1'', \cdots, I_n''$ de longueur $|I_i'| = |I_i''| = \xi_i$ ($i = 1, \cdots, n$), pour lesquels

$$N(\alpha_1, \cdots, \alpha_n, x, I_1', \cdots, I_n') > x \cdot \xi_1\xi_2 \cdots \xi_n,$$

$$N(\alpha_1, \cdots, \alpha_n, x, I_1'', \cdots, I_n'') \leqq x \cdot \xi_1\xi_2 \cdots \xi_n.$$

(B) Si tous les nombres $\xi_1, \xi_2, \cdots, \xi_n$ sont rationnels, on a (I), où le signe $\leqq$ peut être remplacé par $<$, ou bien on a (II) $N(\alpha_1, \cdots, \alpha_n, x, I_1, \cdots, I_n) = x \cdot \xi_1\xi_2 \cdots \xi_n$ quels que soient les intervalles $I_1, \cdots, I_n$ de longueur $\xi_1, \cdots, \xi_n$ respectivement.

J. F. Koksma (Amsterdam).

ANALYSIS

Bonsall, F. F. The characterization of generalized convex functions. Quart. J. Math., Oxford Ser. (2) **1**, 100–111 (1950).

The author studies generalized convex functions in the special case for which the two-parameter family of majorizing functions is given by the solutions of a second order differential equation of the form

$$(*) \qquad L(y)\equiv d^2y/dx^2+p_1(x)dy/dx+p_2(x)y=0;$$

it is supposed that there is a unique solution of (*) taking on any real values y_1 and y_2 at any x_1 and x_2 in the interval (a, b) of definition. Thus it is shown that a sub-$\{L\}$ function y must have a second derivative almost everywhere in (a, b), and must satisfy $L(y)\geqq 0$ at each point where the second derivative exists; conversely, if a function y has a continuous second derivative satisfying $L(y)\geqq 0$, then y is a sub-$\{L\}$ function. These results have been anticipated in part by Matos Peixoto [Bull. Amer. Math. Soc. **55**, 563–572 (1949); these Rev. **10**, 686], who assumed continuous second derivatives throughout, but who on the other hand dealt with nonlinear differential inequalities. Applications are made to determinantal inequalities, to the growth of the mean value function $M_t(a)$, and the approximate solution of differential equations. *E. F. Beckenbach* (Los Angeles, Calif.).

Hilding, Sven H. On the constant in Hölder's inequality. Ark. Mat. **1**, 93–100 (1950).

If we let $I_r=\int_E[f(x)]^r[g(x)]^{1-r}d\varphi(x)$, $0\leqq r\leqq 1$, where $f(x)$ and $g(x)$ are nonnegative functions and $\varphi(x)$ is an increasing function on E, then Hölder's inequality can be written as $I_r\leqq I_1^rI_0^{1-r}$; this expression in terms of Lebesgue-Stieltjes integrals includes the usual inequality for finite sums and also its familiar integral analogue. For Hölder's inequality (as for other fundamental inequalities), we might seek best constants (or functions of certain parameters) $\psi_r\geqq 0$ and $\theta_r\leqq 1$ such that $\psi_rI_1^rI_0^{1-r}\leqq I_r\leqq\theta_rI_1^rI_0^{1-r}$ under suitably prescribed additional restrictions on $f(x)$ and $g(x)$. In studying the problem, there is no essential loss of generality in taking $I_0=I_1=1$; and we shall assume that $f(x)$ and $g(x)$ have been normalized in this way. Since $I_r=1$ only when the normalized functions satisfy $f(x)\equiv g(x)$, it follows that $I_r<1$ if this possibility is excluded. Indeed, if there exist constants a, b, with $a<1<b$, such that $f(x)/g(x)$ takes on no value in (a, b), then [see F. Carlson, Ark. Mat. Astr. Fys. **26A**, no. 9, 1–13 (1938), pp. 4–5] we have

$$(*) \qquad \theta_r=[(1-a)b^r+(b-1)a^r]/(b-a).$$

It is now shown, more generally, that the value of I_r is dominated by an explicit function $H_r(u, v)$ of two parameters (u, v) depending on f, g, and φ, with region of definition the triangle $u+v-1\geqq\theta$, $u\leqq 1$, $v\leqq 1$; further, we have $H_r(u, v)\leqq 1$, the sign of equality holding if and only if $u+v-1=0$; and for each $H_r(u_0, v_0)$ there are functions f, g, and φ for which $I_r=H_r(u_0, v_0)$. Thus a restriction which bars (u, v) from a neighborhood of $u+v-1=0$ yields a value of θ_r which is less than 1. The method is applied to yield (*) and also to treat, for example, the case where $\int_E|f(x)-g(x)|d\varphi(x)\geqq\eta>0$.

For results concerning ψ_r, $r=\frac{1}{2}$, see the following review. *E. F. Beckenbach* (Los Angeles, Calif.).

Bückner, Hans. Untere Schranken für skalare Produkte von Vektoren und für analoge Integralausdrücke. Ann. Mat. Pura Appl. (4) **28**, 237–261 (1949).

For $r=\frac{1}{2}$, the function ψ_r [see the preceding review] can be given [Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, v. 1, Springer, Berlin, 1925, p. 57] in terms of the maxima A, B and minima a, b of f, g, respectively, by $\psi_{\frac{1}{2}}=2/[(AB/ab)^{\frac{1}{2}}+(ab/AB)^{\frac{1}{2}}]$. For this case, the author continues the work of Blaschke and Pick [Math. Ann. **77**, 277–300 (1916)], who showed that for all positive continuous convex functions f, g we have $\psi_{\frac{1}{2}}=\frac{1}{2}$. Thus he shows that for all positive nondecreasing convex functions f, g we have $\psi_{\frac{1}{2}}=\frac{1}{2}\sqrt{3}$. His principal result is the following. If $u(s)$, $K(s, t)$, and $\varphi(s)$ are given, with $\varphi(s)$ monotonic, and we set $v(s)=\int_0^1K(s, t)dh(t)$, with $h(t)$ monotonic, and let $T(t)=[\int_0^1u(s)K(s, t)d\varphi(s)]/[\int_0^1K^2(s, t)d\varphi(s)]^{\frac{1}{2}}$, and $T_0=\min_{0\leqq t\leqq 1}|T(t)|=T(t_0)$, then we have

$$\left|\int_0^1u(s)v(s)d\varphi(s)\right|\geqq T_0\cdot\left[\int_0^1v^2(s)d\varphi(s)\right]^{\frac{1}{2}};$$

the equality sign holds if and only if $v(s)=K(s, t_0)$.

E. F. Beckenbach (Los Angeles, Calif.).

Nikol'skiĭ, S. M. Concerning estimation for approximate quadrature formulas. Uspehi Matem. Nauk (N.S.) **5**, no. 2(36), 165–177 (1950). (Russian)

Let $f(x)$ be a function defined in (a, b) and $x_0, x_1, \cdots, x_n$ and $p_0, p_1, \cdots, p_n$ be given numbers. In the computation of the integral $\int_a^bf(x)dx$ by the approximate quadratures $\sum_0^np_kf(x_k)$ we may be interested not only in the error for individual functions f but also in the maximum of the error for all f belonging to a certain class H of functions. Correspondingly, we set

$$E_n(H; p_k, x_k)=\sup_{f\in H}\left|\int_a^bf(x)dx-\sum_0^np_kf(x_k)\right|.$$

The author investigates the cases when H is one of the classes $KW^{(r)}M$, $KW^{(r)}L^p$, $KW_0^{(r)}M$, $KW_0^{(r)}L^p$. The first two classes are those of the functions f differentiable r times and satisfying respectively the conditions $|f^{(r)}|\leqq K$, $(\int_a^b|f^{(r)}|^pdx)^{1/p}\leqq K$ $(p\geqq 1)$. In the remaining two cases we add the conditions $f(a)=f'(a)=\cdots=f^{(r-1)}(a)=0$. Numerical data for $r=1, 2, 3, 4$ are given when the x_k and p_k are those of the trapezoidal formula and the formula of Simpson. In the second part of the paper the numbers x_k and p_k are no longer given but, for a fixed class H of functions, are obtained in such a way that $E_n[H]=E_n(H, a, b)$ is a minimum. The case $r=2$ is investigated in greater detail and, e.g., for $H=W_0^{(2)}M$ the author obtains the x_k so that the approximation is 1.5 times better than given by Simpson's formula for the same number of nodal points.

A. Zygmund (Chicago, Ill.).

Meyers, Leroy F., and Sard, Arthur. Best approximate integration formulas. J. Math. Physics **29**, 118–123 (1950).

On désigne par $a_{m,n}$ l'ensemble des formules:

$$c_0x(0)+c_1x(1)+\cdots+c_mx(m)$$

qui donnent la valeur exacte de $\int_0^mx(t)dt$ lorsque $x(t)$ est un polynome de degré n. Lorsque $x(t)$ est une fonction possédant une dérivée $(n+1)$-éme continue, l'erreur résultant de l'emploi de cette formula peut s'écrire: $\int_0^mx^{(n+1)}(t)k(t)dt$,

$k(t)$ ne dépendant pas de la fonction $x(t)$. Les auteurs appellent meilleure formule parmi les formules $a_{m,n}$ celle pour laquelle $J=\int_0^m k^2(t)dt$ est minimum. L'inégalité de Schwarz donne alors comme borne supérieure de l'erreur commise dans l'emploi de la formule: $[J\int_0^m[x^{(n+1)}(t)]^2dt]^{\frac{1}{2}}$. Les auteurs donnent les coefficients c_i et la valeur de J pour $n=1$, $m\leqq 20$; $n=2$, $m\leqq 12$; $n=3$, $m\leqq 9$. Ils indiquent quelques propriétés de ces coefficients et en suggèrent d'autres qui ne sont pas démontrées. *J. Kuntzmann.*

Sard, Arthur. Smoothest approximation formulas. Ann. Math. Statistics **20**, 612–615 (1949).

Consider $A=\int x(t)d\alpha$, an approximation to the functional $f(x)$. Write $R=A-f(x)$. Suppose x replaced by $x+\delta x$ where $x(t)$ is a stochastic process with mean 0. Then $R+\delta A=A+\delta A-f(x)$ where δA can be interpreted as the error due to uncertainty in x. The problem is to select α from a family of possible choices. The author defines smoothest approximation by the requirement (on the choice of α) that the variance of δA be minimized, if possible. As an application he treats $f(x)=\int_{-1}^{1}x(t)dt$ where it is demanded that $\alpha(t)$ be a step function with equi-spaced jumps and that the approximation be exact for all nth degree polynomials. *D. G. Bourgin* (Urbana, Ill.).

Milne, W. E. The remainder in linear methods of approximation. J. Research Nat. Bur. Standards **43**, 501–511 (1949).

This paper may be recast and generalized as follows. Write E and E^* for a Banach space and its conjugate. Let E_n be the linear extension of $\{x^i|\ i=1,\cdots,n\}$, and let $\{f^i|\ i=1,\cdots,n\}$ be elements of E^* which are linearly independent over E. For f in E^* write (1) $R(x)=f(x-\sum_1^n A_jx^j)$ subject to (2) $f^i(x-\sum_1^n A_jx^j)=0$, $i=1,\cdots,n$. Then (2) determines $\{A_j\}$ linearly in terms of $\{f_j(x)\}$ and $\{f_i(x^j)\}$. Hence (1) and (2) can be replaced by the statement R is a given element of E^*. Clearly, (3) $R(z)=0$ for $z\varepsilon E_n$. Let U be a linear operator on E to E' whose null space is finite dimensional; for instance, we may require that $I-U$ be completely continuous. The $\{x^i\}$ and E_n refer to the null space of U. Let U' be the transformation induced by U on E/E_n to E', and M' its inverse. Then, with the obvious interpretation of M, we have $Ux=y$ implies $x=z+My$ where $z\varepsilon E_n$, and hence, using (3), there results the identity (4) $R(x)=RMU(x)$. Suppose now that the spaces are function spaces with functionals and operators admitting representations in terms of suitably defined integrals and measures. Then $R(x)=\int_A x(t)dr(t)$, and if $R'y=Rx$ for $y=U(x)$ then (5) $R'U(x)=\int_B Ux(s)dH(s)$. If $(My)(t)=\int_C y(s)d_sK(t,s)$ then (4) yields information about the function $H(s)$ occurring in (5), since, by writing down the integrals involved in the right side of (4) and changing the order of integration, one arrives at $H(s)=\int_D K(t,s)dr(t)$. The case of variable limits is taken care of by requiring the functions to vanish on suitable subsets. The interest is in bounding $\|R(x)\|$. Now (6) $\|R(x)\|=\|\int_B(Ux)(s)dH(s)\|\leqq\|Ux\|\|H\|$. The conditions under which $H(s)$ is of such a nature that this bound is close are significant. The paper concerns itself with the special case of the formulation above of real functions of one real variable with $E=C^n(a,b)$, $E'=C(a,b)$, and U a differential operator of order n. Here R has the Riesz representation as a Riemann-Stieltjes integral and $dH(s)=G(s)ds$. The author discusses the case $U=d^n/ds^n$. Here $x^i(s)=s^{i-1}$, $G(s)$ is a polynomial (whose qualitative features are noted in some detail), and $r(s)$ is a step function. The formulae thus obtained are interpreted as numerical approximation expressions for interpolation and approximate integration with $R(x)$ as the error and include the usual standard formulae and many new ones. There is particular interest in the case where $G(s)$ is of one sign, for then, in the notation of (6), $\|H\|=|\int G(s)ds|$. *D. G. Bourgin* (Urbana, Ill.).

Theory of Sets, Theory of functions of Real Variables

Einführung in die reelle Analysis

***Haupt, Otto, Aumann, Georg, and Pauc, Christian. Differential- und Integralrechnung unter besonderer Berücksichtigung neueren Ergebnisse.** Band I. 2d ed. Walter de Gruyter & Co., Berlin, 1948. viii+218 pp. 18 DM.

This is an introduction to the theory of real functions, with some modern improvements. In spite of the title, it contains no differential calculus, and very little integral calculus. A first edition by Haupt and Aumann appeared in 1938 [de Gruyter, Berlin]; the present revision is by Haupt and Pauc, and is the first volume of a projected larger work.

Chapter titles are: real numbers, limits of sequences, the function concept, real functions of a real variable, of several real variables, the general theory of limits. The following are some noteworthy topics included: the well-ordering theorem, transfinite induction (but not Zorn's lemma), limits of sets, Borel and Suslin sets, directed systems and Moore-Smith convergence, length of arc, convex and biconvex functions, convex mean values, subdivision integrals and the Burkill integral, interchange of order of limiting processes, abstract topological spaces, including separation axioms and countability axioms, functions with values in a topological space. There is no theory of measure or of additive functions of sets. *O. Frink* (State College, Pa.).

Lorent, H. Un postulat implicite de la théorie des ensembles. Bull. Soc. Roy. Sci. Liège **19**, 194–196 (1950).

A principle concerning countable subsets of a set of cardinal number $2^{\aleph_0}$. This principle is obviously false, and in any case seems irrelevant to the existence of the Riemann integral, which the author considers. *E. Hewitt.*

Sierpiński, W. Sur la décomposition des ensembles en paires. Proc. Benares Math. Soc. (N.S.) **8**, no. 2, 31–34 (1946).

If a nonfinite set E is decomposed into a family F of disjoint sets, each consisting of two elements, then F has the same power as E; a theorem first proved using the axioms of choice, is demonstrated here without this axiom. *J. F. Randolph* (Rochester, N. Y.).

Sierpiński, W. Opérations sur les familles d'ensembles. Proc. Benares Math. Soc. (N.S.) **9**, no. 1, 1–24 (1947).

An operation on families of sets is any rule which assigns to every family of sets Φ another family of sets Φ_γ, where the family Φ_γ is completely determined by the family Φ. This paper, which is largely expository, summarizes the principal results which have been obtained concerning such operations, many of them by the author himself. *E. Hewitt* (Seattle, Wash.).

Aleksandrov, P. S. On ordered systems of closed and open sets. Uspehi Matem. Nauk (N.S.) **5**, no. 2(36), 178–179 (1950). (Russian)

A completely ordered set Θ is said to be a simply ordered set if every $\alpha \varepsilon \Theta$ has an immediate successor. Lemma. Every family of sets of natural numbers with a simple ordering under inclusion is countable. Theorem. In a topological space with a countable basis, every system of pair-wise distinct open (closed) sets having a simple ordering under inclusion is countable. The proofs, which are trivial, are offered as being acceptable in courses where transfinite induction is not assumed. *E. Hewitt* (Seattle, Wash.).

Eggleston, H. G. A geometrical property of sets of fractional dimension. Quart. J. Math., Oxford Ser. (2) **1**, 81–85 (1950).

A theorem of product-measure type. Let A be a plane set of s-measure $\Lambda_s(A)$. Each point $p \varepsilon A$ is an endpoint of an arc $L(p)$ of length l and such that, for some k in $0<k\leqq 1$, $\rho(x_1, x_2) \geqq k\rho(p_1, p_2)$ for all $x_1 \varepsilon L(p_1)$, $x_2 \varepsilon L(p_2)$. Then the set $L(A)=\sum L(p)$ has outer $(s+1)$-measure at least $k^s l \Lambda_s(A)$. The author remarks that the arcs may be replaced by regular linear sets. *H. D. Ursell* (Leeds).

Sikorski, Roman. Closure algebras. Fund. Math. **36**, 165–206 (1949).

This paper treats σ-complete Boolean algebras A in which there is defined a closure operation satisfying, formally, Kuratowski's axioms. Examples are $S(X)$ and $B(X)$, where these are the classes of all, and of Borel, subsets of a topological space. One may ask whether an A can be represented as an $S(X)$ or whether the subclass $B(A)$ of "Borel elements" in A can be represented by some $B(X)$. Sometimes neither is possible. Special attention is given to C-algebras A, in which there exist a sequence of open elements $R_1, R_2, \cdots$ such that every open G is the sum of those R_n such that $R_n^- \leqq G$. It is shown that if A is a C-algebra, then $B(A)$ is isomorphic to $B(S(X)/I)$ where X is a separable metric space and I is a σ-ideal in $S(X)$. The problem of defining convergence for sequences of homeomorphisms of C-algebras is also treated. *R. Arens* (Los Angeles, Calif.).

Cotlar, M., and Ricabarra, R. A. On a theorem of E. Hopf. Revista Unión Mat. Argentina **14**, 49–63 (1949). (Spanish)

Let M be a set, T a Borel field of subsets of M, m a positive countably additive measure on T with $m(M)=1$, G a transformation group on M preserving T and absolutely continuous with respect to m. Under the assumption that G is "measurable" (in the sense that G admits some positive right-invariant additive measure defined on the collection of all its subsets giving G the measure 1), the authors establish that the following conditions are equivalent. (a) There exists a positive countably additive measure μ on T which is $(m-G)$-invariant, that is, μ is absolutely continuous with respect to m, $\mu(gA)=\mu(A)$ for $g \varepsilon G$, $A \varepsilon T$, and $\mu(M)=1$. (b) No set $M' \varepsilon T$ with $m(M')<m(M)$ is equivalent to M by infinite decompositions, that is for such an M' the decompositions $M=\bigcup_{n=1}^{\infty} A_n$, $M'=\bigcup_{n=1}^{\infty} A_n'$ with $A_n'=g_n A_n$, $A_n \varepsilon T$, $g_n \varepsilon G$, $A_i \cap A_j = A_i' \cap A_j' = \phi$ $(i \neq j)$ are impossible. (c) G is equicontinuous in the sense that corresponding to $\epsilon>0$ there exists $\delta>0$ such that $A \varepsilon T$, $g \varepsilon G$, $m(A)<\delta$ imply $m(gA)<\epsilon$.

The equivalence between (a) and (b) was proved by E. Hopf [Trans. Amer. Math. Soc. **34**, 373–393 (1932)] when G is the additive group of the integers or that of the reals. The authors also study the "singular nucleous," a certain subset of M explaining completely the existence or not of a $(m-G)$-invariant measure, and make use of an example due to Halmos [Ann. of Math. (2) **48**, 735–754 (1947); these Rev. **9**, 137, 735] to point out some pathological properties of the additive group of the integers. *L. Nachbin.*

Ionescu Tulcea, C. T. Mesures dans les espaces produits. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7** (1949), 208–211 (1950).

Let $E=\prod_{s \varepsilon K} X_s$ be an infinite product space (K not necessarily countable). For each $s \varepsilon K$, let $\mathfrak{B}_s$ be a σ-field of subsets of X_s. For each finite subset I of K, let $\mathfrak{B}^I$ be the σ-field of subsets of E determined by $\mathfrak{B}_s$, $s \varepsilon I$. Let $\mathfrak{F}$ be the union of $\mathfrak{B}^I$ for all finite subsets I of K. Then, $\mathfrak{F}$ is a field, but, in general, is not a σ-field. The problem is: given a finitely additive measure P defined on $\mathfrak{F}$ which is countably additive on each $\mathfrak{B}^I$, is it possible to extend P to a countably additive measure defined on the σ-field generated by $\mathfrak{F}$? [Cf. J. L. Doob, Trans. Amer. Math. Soc. **44**, 87–150 (1938), pp. 90–93.] It is known that this extension is possible if P is a direct product measure. As was shown by Andersen and Jessen [Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 4 (1948), pp. 4–5; these Rev. **10**, 239] and Dieudonné [cf. P. Halmos, Measure Theory, van Nostrand, New York, 1950, p. 214, problem 49(3); these Rev. **11**, 504], this extension is not always possible if P is not a direct product measure. The author proves that the extension is possible if there exists, for any finite subset $I=\{s_1, \cdots, s_n\} \subset K$ and for any $t \varepsilon K-I$, a conditional probability function $P(x_{s_1}, \cdots, x_{s_n}; A_t)$ in the sense of Doob [op. cit., pp. 95–103] $(x_{s_i} \varepsilon X_{s_i},\ i=1, \cdots, n;\ A_t \varepsilon B_t)$. *S. Kakutani.*

Pitt, H. R. The definition of measure in function space. Proc. Cambridge Philos. Soc. **46**, 19–27 (1950).

A new attempt is made to introduce a probability measure on the space Ω of all real-valued functions $x(t)$ of a real variable t. Kolmogoroff [Grundbegriffe der Wahrscheinlichkeitsrechnung, Springer, Berlin, 1933] discussed measures defined on the σ-field $\mathfrak{B}$ generated by sets of the form: $E(t, a, b)=\{x \mid a<x(t)<b\}$. This σ-field $\mathfrak{B}$ is not large enough for a decent theory of stochastic processes with a continuous parameter t. For example, $\mathfrak{B}$ does not contain a set of the form: $E(I, a, b)=\{x \mid a<x(t)<b \text{ for all } t \varepsilon I\}$, where I is any open or closed interval of real numbers t. Further, the function $x(t)$ is not measurable as a function of two variables x and t. Doob's [Trans. Amer. Math. Soc. **42**, 107–140 (1937)] method of taking a nonmeasurable subset Ω_0 of Ω of outer measure 1 and of considering the induced measure on Ω_0 solved the first difficulty, but the second difficulty is not yet completely solved. The author considers measures defined on a larger σ-field $\mathfrak{B}'$ generated by sets $E(I, a, b)$, where I is any open or closed interval or any one-point set of real numbers. The author discusses conditions on a nonnegative function $M(I_1, \cdots, I_n; a_1, \cdots, a_n; b_1, \cdots, b_n)$ in order that there exist a countably additive measure m defined on $\mathfrak{B}'$ such that

$$m\left(\bigcap_{i=1}^{n} E(I_i, a_i, b_i)\right) = M(I_1, \cdots, I_n; a_1, \cdots, a_n; b_1, \cdots, b_n)$$

for all I_i, a_i, b_i, $i=1, \cdots, n$; $n=1, 2, \cdots$. The author communicates that theorem 3 of the present paper is incorrect. An unsolved problem, which the author mentions, is the investigation of the relation between such measures on

$\mathfrak{B}'$ and regular topological measures defined on a still larger σ-field $\mathfrak{B}''$ consisting of all Borel sets of subsets of Ω with respect to the weak topology (= product topology) of Ω which were discussed by the reviewer [Proc. Imp. Acad. Tokyo **19**, 184–188 (1943); these Rev. **7**, 279] and Doob [Bull. Amer. Math. Soc. **53**, 15–30 (1947); these Rev. **8**, 472].

S. Kakutani (New Haven, Conn.).

Sibirani, Filippo. Sulle funzioni ordinatrici. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (10) **5** (1947/48), 27–34 (1949).

Given a real-valued measurable function on $a \leqq x \leqq b$, let f_1 and f_2 be the nondecreasing and nonincreasing functions on $a \leqq x \leqq b$ equimeasurable with f. The author gives a number of examples to show how properties of f determine corresponding properties of the f_i. *M. M. Day.*

Gál, I. S., et Koksma, J. F. Sur l'ordre de grandeur des fonctions sommables. Nederl. Akad. Wetensch., Proc. **53**, 638–653 = Indagationes Math. **12**, 192–207 (1950).

Let $F_1(x), F_2(x), \cdots$ be a sequence of functions defined on $(0, 1)$ and of the class L^p, $p>1$. Suppose we have an inequality

$$\int_0^1 |F_m - F_n|^p dx = O\{\varphi(m, n)\},$$

where $\varphi(m, n)$ satisfies certain conditions. What can be said about the order of magnitude of $F_n(x)$ almost everywhere as $n \to \infty$? This problem occurs comparatively often, and in a number of cases ingenious devices have been invented to obtain sharp solutions (for the case when F_n is the partial sum of an orthogonal development see, for example, the papers of Menchoff [Fund. Math. **4**, 82–105 (1923)] and Rademacher [Math. Ann. **87**, 112–138 (1922)]). Here the authors systematize some such devices and give them abstract forms. The formulations are too long to be given here.

A. Zygmund (Chicago, Ill.).

Koksma, J. F. An arithmetical property of some summable functions. Nederl. Akad. Wetensch., Proc. **53**, 959–972 = Indagationes Math. **12**, 354–367 (1950).

Let $g(x) \in L^2$ have period one and be such that, if S_m denotes the mth partial sum of its Fourier series, one has $\sum m^{-1} \int_0^1 (g - S_m)^2 dx < \infty$. Let $u_n(x)$, $n=1, 2, \cdots$, be a sequence of functions defined for $0 \leqq x \leqq 1$. The author proves that the relation

$$\lim_{n \to \infty} [g(u_1(x)) + \cdots + g(u_n(x))]/n = \int_0^1 g(t)dt$$

holds almost everywhere in x for a wide class of sequences $u_n(x)$, among which one can quote in particular $u_n(x) = nx$, $u_n(x) = \lambda_n x$ $(\lambda_{n+1} - \lambda_n \geqq \delta > 0)$, $u_n(x) = (1+x)^n$.

R. Salem (Cambridge, Mass.).

Sierpiński, W., and Singh, A. N. On derivates of discontinuous functions. Fund. Math. **36**, 283–287 (1949).

The authors prove the following results. (i) There exists a function $f(x)$, continuous on the right but discontinuous at an everywhere dense set, such that everywhere the lower right derivate of $f(x)$ is zero. (The construction depends on the expression of $f(x)$ in the scale of 3 defined in terms of the expression of x in the scale of 2.) (ii) If a function has a finite right hand differential coefficient everywhere, the discontinuities of the function form a nondense set. The use of the symbol $f'_+(x)$ to denote both the lower right derivate and also (in the latter part of the paper) the right hand differential coefficient may cause some trouble to the reader.

U. S. Haslam-Jones (Oxford).

Cesari, Lamberto. Condizioni necessarie per la semicontinuità degli integrali sopra una superficie in forma parametrica. Ann. Mat. Pura Appl. (4) **29**, 199–224 (1949).

Let A be a closed bounded subset of xyz-space, and let $F(x, y, z, u, v, w)$ be a (real-valued) function with the following properties. (i) F is continuous for $(x, y, z) \in A$ and $u^2+v^2+w^2>0$. (ii) F is positively homogeneous of degree 1 in the variables u, v, w. (iii) For the arguments described under (i), F has continuous first partial derivatives F_u, F_v, F_w. Let now $S: x=x(u, v), y=y(u, v), z=z(u, v), (u, v) \in Q: 0 \leqq u \leqq 1, 0 \leqq v \leqq 1$, be a Fréchet surface with finite Lebesgue area $L(S)$. For such surfaces, the author considers the integral of F extended over S, in a sense introduced by him. For the present paper, the following fact concerning this integral I_S is fundamental. If the representation of S is absolutely continuous (in the sense in which this term is used in the theory of the Lebesgue area), then I_S is equal to the customary variational integral, as a Lebesgue integral, calculated in terms of generalized Jacobians. Let us consider the class K of those surfaces S whose points are comprised in A, and which have finite Lebesgue area. Let $S_0 \in K$, and consider an absolutely continuous representation of S_0. For such a representation the Weierstrass E-function may be defined by the usual formula. Assume now that the integral I_S is lower semicontinuous at S_0 in the class K. The author proves that $E \geqq 0$ at almost every point (u, v) where the three generalized Jacobians of S_0 do not vanish simultaneously. The validity of this theorem is further evidence of the appropriateness of the basic concepts involved in the definition of the integral I_S. *T. Radó* (Columbus, Ohio).

Cesari, Lamberto. Sul problema di Geöcze. Rivista Mat. Univ. Parma **1**, 207–227 (1950).

Let $x(u, v)$ denote a continuous function with values in 3-space and with arguments in a simple plane Jordan domain J, and let $x_n(u, v)$ denote, for each $n=1, 2, \cdots$, a function defined on a polyhedral subdomain J_n of J such that $x_n(u, v)$ is linear in each triangle of a certain triangulation T_n of J_n and that $x_n(u, v) = x(u, v)$ at the vertices of T_n. Further let S, S_n, $a(S)$, $a(S_n)$ denote respectively the Fréchet surfaces defined by $x(u, v)$ on J and by $x_n(u, v)$ on J_n, and their Lebesgue areas. The sequence $\{x_n(u, v)\}$ will be here termed a Geöcze sequence of inscribed polyhedra for $x(u, v)$ if $S_n \to S$ and $a(S_n) \to a(S)$. The problem of the existence of such a sequence when $x(u, v)$ is given, is termed the Geöcze problem for $x(u, v)$, or strong Geöcze problem. The problem of its existence for some $x(u, v)$, when S is given, is termed the Geöcze problem for S, or weak Geöcze problem. Mambriani [Ann. Scuola Norm. Super. Pisa (2) **13** (1944), 1–17 (1948); these Rev. **9**, 417] solved this strong Geöcze problem for a Dirichlet representation $x(u, v)$, i.e., for an absolutely continuous $x(u, v)$ whose partial derivatives have summable squares. This provides a solution of the weak Geöcze problem, in particular, for surfaces quasi-conformally representable on the disc, and so for nondegenerate surfaces according to the well-known theorems of C. B. Morrey. The machinery devised by Cesari [Ann. Mat. Pura Appl. (4) **26**, 301–374 (1947); these Rev. **10**, 362] in his study of quasi-conformal representation, now provides, with the help of an extension of Mambriani's method, a solution of the weak Geöcze problem for an arbitrary continuous surface S.

The reader should note that the notion of inscribed polyhedron S of a surface S, used by Cesari and in the present review, is less stringent than usual in that the boundary of S_n is not restricted to be inscribed in the boundary of S. Besides the strong Geöcze problem it would seem that there still remain to solve the modifications of both problems which arise when inscribed polyhedra are understood in the more usual sense. The sharper form of the strong Geöcze problem which arises in this way has been solved for a non-parametric representation $z=f(x, y)$ by Mulholland [Trans. Amer. Math. Soc. 68, 330–336 (1950); these Rev. 11, 240] and Verčenko [Doklady Akad. Nauk SSSR (N.S.) 68, 5–8 (1949); these Rev. 11, 240]. *L. C. Young.*

Cesari, Lamberto. Area and representation of surfaces. Bull. Amer. Math. Soc. 56, 218–232 (1950).

This paper contains the text of an invited address, delivered by the author at the annual meeting of the American Mathematical Society in December, 1948. The object is to give a picture of the status of the theory of the Lebesgue area of surfaces, with particular emphasis upon relationships to the calculus of variations, as regards both motivation and application. In the theory of single integral problems in the calculus of variations the concept of semicontinuity, the analytic concepts of derivative, integral, bounded variation, absolute continuity, and the geometric concepts of curve and arc length played a fundamental role. During the past fifty years, a great deal of effort was expended to develop analogous two-dimensional concepts of comparable scope and utility. The author gives an account of the development of this program in terms of the Lebesgue definition of surface area. *T. Radó* (Columbus, Ohio).

Cecconi, Jaurès. Su le funzioni caratteristiche e gli Jacobiani generalizzati. I. Su la nozione di "molteplicità relativa." Rivista Mat. Univ. Parma 1, 229–235 (1950).

Let $A: 0\leqq u\leqq 1,\ 0\leqq v\leqq 1$, be the unit square in the (u, v)-plane, and let $T: x=x(u, v),\ y=y(u, v)$ be a continuous mapping from A into the (x, y)-plane. Previously the reviewer and Reichelderfer in the United States and Cesari in Italy developed, independently, comprehensive theories for such mappings for the purpose of applications in surface area theory and in the calculus of variations. An excellent presentation of these theories may be found in an expository paper of Cesari [see the preceding review] which also contains an extensive bibliography. The basic concepts in these theories are two-dimensional concepts of bounded variation, absolute continuity, and generalized Jacobian. Recently, the relationship between these two theories became a subject of systematic study. Completing previous results, the author establishes the equivalence (almost everywhere) of the concepts of the signed (or relative) multiplicity functions occurring in the two theories, and states that in a subsequent paper he will prove the corresponding result for generalized Jacobians. *T. Radó* (Columbus, Ohio).

Valiron, Georges. The notions of area and volume. An. Soc. Ci. Argentina 149, 185–195 (1950). (Spanish)

Tolstov, G. P. On change of variables in multiple integrals. Uspehi Matem. Nauk (N.S.) 5, no. 4(38), 162–169 (1950). (Russian)

Let Γ be a piecewise smooth simple contour in the (u, v)-plane and let D be the closure of its interior domain. Let $(x, y)=T(u, v)=[\varphi(u, v), \psi(u, v)]$ be continuous in a domain containing D and possess continuous partial derivatives in D. Suppose further that T is one-to-one on Γ. Write J for the Jacobian of T, C for $T(\Gamma)$, and G for the closure of the interior domain of C [so that G is contained in $T(D)$]. Then for any function $f(x, y)$ summable in $T(D)$ we have $\iint_G f dxdy=\delta\iint_D f(\varphi, \psi)Jdudv$, where $\delta=\pm 1$ is the sign of $\iint_D Jdudv$. As the author states, this is no new result.

L. C. Young (Madison, Wis.).

Theory of Functions of Complex Variables

***Fuks, B. A., and Šabat, B. V. Funkcii kompleksnogo peremennogo i nekotorye ih priloženiya. [Functions of a Complex Variable and Some of Their Applications].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 383 pp.

This is a rather elementary introductory text which does not discuss theoretical details, but treats in a very detailed manner such topics as evaluation of integrals and special conformal mappings, with a profusion of applications to problems in electrostatics, hydrodynamics and heat flow.

R. P. Boas, Jr. (Evanston, Ill.).

***Markušević, A. I. Teoriya analitičeskih funkcii. [Theory of Analytic Functions].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 703 pp.

This very detailed treatise includes, in addition to the topics customary in an introductory course in the subject, brief accounts of such topics as the theory of univalent functions, conformal mapping at the boundary, polynomial interpolation and approximation, entire and meromorphic functions, and modular functions, Picard's theorem, and Julia lines. *R. P. Boas, Jr.* (Evanston, Ill.).

Markouchevitch, A. Quelques remarques sur les intégrales du type de Cauchy. Učenye Zapiski Moskov. Gos. Univ. 100, Matematika, Tom I, 31–33 (1946). (Russian. French summary)

In terms of coefficients of a Taylor series and of Fourier coefficients the author obtains necessary and sufficient conditions for: (a) the function $f(z)=\sum_0^\infty a_n z^n$ ($|z|<1$) to be expressible by an integral of Cauchy type; (b) the integral of Cauchy type $I(z)=(1/2\pi i)\int_{|\zeta|=1}(\zeta-z)^{-1}w(\zeta)d\zeta$ ($|z|<1$) to be expressible by a Cauchy integral.

W. J. Trjitzinsky (Urbana, Ill.).

Markushevitch, A. On a boundary problem of the theory of analytic functions. Učenye Zapiski Moskov. Gos. Univ. 100, Matematika, Tom I, 20–30 (1946). (Russian, English summary)

With γ a closed rectifiable Jordan curve, g and G its interior and exterior domains, respectively, the function $f(z)$ [$F(z)$] is said to be of class E_1 in g [in G] if it is analytic in the domain and the integral $\int_{c_r}|f|ds$ [$\int_{c_r}|F(z)-F(\infty)|ds$] is uniformly bounded for $0<r<1$; here c_r [C_r] is the homologue of the circumference $|w|=r$ under the conformal mapping of g [G] on the domain $|w|<1$; $f(z)$ of E_1 (in g) satisfies Cisotti's condition in g, if for some $F(z)$ of E_1 (in G) one has $F(\zeta)=f(\zeta)$ almost everywhere on γ; this condition is equivalent to the representation

$$(1)\qquad f(z)=f(0)+(1/\pi i)\int_\gamma\{z\Re f(\zeta)/\zeta(\zeta-z)\}d\zeta.$$

It is shown that if a univalent, continuous function satisfies Cisotti's condition for g, then γ is a circle and that (1),

applied to an arbitrary function on g, yields not more than the Schwartz formula. The case, as related to Cisotti's condition, is examined when γ is a generalized lemniscate $|(A_0+\cdots+A_p z^p)/(B_0+\cdots+B_p z^p)|=1$.

W. J. Trjitzinsky (Urbana, Ill.).

Consiglio, Alfonso. Su la risoluzione in termini finiti dei problemi di Villat e Dirichlet relativi alla corono circolare. Atti Accad. Gioenia Catania (6) **6** (1943/49), no. 7, 14 pp. (1950).

L'auteur, complétant une note antérieure [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **26**, 66–75 (1937)], étudie la formule de Villat exprimant la fonction, holomorphe dans une couronne, qui prend sur les circonférences limitantes des valeurs données de même moyenne. Lorsque ces données sont développables en séries de Fourier, la formule peut se transformer de façon à ne plus contenir d'intégrale mais seulement les coefficients des développements. En général on peut lui donner une forme analytique se prêtant au calcul des résidus. Tout cela évidemment résoud implicitement de même le problème de Dirichlet pour la couronne. *M. Brelot* (Grenoble).

*Carleman, Torsten. **Application de la théorie des fonctions analytiques à la résolution de certaines équations fonctionnelles.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 209–221, Rome, 1943.

The author gives examples of problems in mathematical physics, which reduce to finding an analytic function, satisfying functional equations on the boundary of its domain of existence. One problem is solved in detail: to determine the equilibrium position of a flexible infinite cylindrical conducting surface with an electric charge distribution which is constant along the generating lines. It is proved that in the equilibrium position the cross section C is a straight-line segment if the curve C is not closed, and is a circle if C is closed. The author solves the related problem of finding all closed curves C for which the electrostatic charge density is proportional to a power of the radius of curvature. In conclusion the author observes that the applications of the theory of analytic functions are not restricted to problems involving the Laplace equation, as the solution of any linear elliptic equation in two independent variables can be expressed in terms of an arbitrary analytic function.

F. John (Los Angeles, Calif.).

Zygmund, A. On a theorem of Littlewood. Summa Brasil. Math. **2**, no. 5, 51–57 (1949).

L'auteur donne deux démonstrations élémentaires d'un théorème de Littlewood [J. London Math. Soc. **2**, 172–176 (1927)]: Si C_0 est une courbe fermée simple, passant par $z=1$, intérieure, sauf ce point, au cercle $|z|<1$, et si $C(\theta)$ est la transformée de C_0 par la rotation $(0,\theta)$, il existe une fonction $f(t)$ bornée, définie sur la circonférence $\Gamma=z=e^{it}$, $0\leqq t<2\pi$, et telle que pour presque tout θ_0, l'intégrale de Poisson

$$(1)\qquad f(r,\theta)=\int_{-\pi}^{\pi} f(\theta+t)P(r,\theta)\,dt,$$

$$P(r,\theta)=(2\pi)^{-1}(1-r^2)/(1-2r\cos t+r^2),\qquad 0\leqq r<1,$$

n'ait pas de limite quand $re^{i\theta}\to e^{i\theta_0}$ le long d'un au moins des arcs de $C(\theta_0)$ qui aboutissent à $e^{i\theta_0}$. La première démonstration utilise la construction d'un produit de Blaschke $\varphi(z)$ dont le zéros, dans $|z|<1$, sont denses sur la circonférence Γ. Soit L_0 un continu aboutissant à $z=1$, intérieur à $|z|<1$, et tangent à P en $z=1$, $L(\theta)$ le continu déduit de L_0 par la rotation $(0,\theta)$. La deuxiéme démonstration construit un ensemble parfait A sur Γ; $f(t)$ étant la fonction caractéristique du complémentaire B de A sur Γ, on a à partir de (1), et pour presque tout $\theta_0\varepsilon A$: lim sup $f(r,\theta)\geqq 1/\pi$, lim inf $f(r,\theta)=0$ quand $z=re^{i\theta}\to e^{i\theta_0}$ selon le chemin $L(\theta_0)$.

P. Lelong (Paris).

Blambert, Maurice. Quelques théorèmes de composition des singularités des séries de Dirichlet. C. R. Acad. Sci. Paris **230**, 1565–1567 (1950).

L'auteur indique quelques cas où les points de la forme $\alpha+\beta$, où α est une singularité de $\sum a_n e^{-\lambda_n s}$ et où β est une singularité de $\sum b_n e^{-\mu_n s}$, est effectivement une singularité de la série composée selon la méthode de Mandelbrojt.

S. Mandelbrojt (Houston, Tex.).

Yu, Chia-Yung. Sur les théorèmes de M. Mandelbrojt du type de M. Hadamard dans la théorie des séries de Dirichlet. C. R. Acad. Sci. Paris **230**, 1490–1491 (1950).

En s'appuyant sur le théorème de Schottky-Valiron, l'auteur généralise le théorème de Mandelbrojt sur la composition des singularités des séries de Dirichlet. Il supprime l'hypothèse que les séries sont bornées ("aux singularités près"), tout en augmentant l'ensemble des singularités possibles de la série composée. *S. Mandelbrojt.*

Sunyer Balaguer, F. On the exclusion of an exceptional function by a gap condition. Mem. Real Acad. Ci. Art. Barcelona (3) **29**, 475–516 (1948). (Spanish)

The author proves and extends to meromorphic functions those results previously announced for integral functions [C. R. Acad. Sci. Paris **224**, 1609–1610 (1947); **225**, 21–23 (1947); these Rev. **9**, 22] in which the hypothesis relates to maximal density. The proof depends on two lines of analysis. If $f(z)=\sum_0^\infty a_n z^{\lambda_n}$ is an integral function of finite order and the maximal density of the λ_n is less than $\gamma/2\pi$ then in each sector subtending an angle γ at the origin the annulus $r(1-\epsilon)<|z|<r(1+\epsilon)$ contains a point z' such that $\log|f(z')|>(1-\epsilon)\log M(r,f)$ for all sufficiently large r [cf. Pólya, Math. Z. **29**, 549–640 (1928), theorem IV, p. 622]. The behaviour of $f(z)$ is then further analysed by means of a well-known lemma of V. Bernstein.

Secondly, Wiman's theory of integral functions of integral order and defective zeros [Ark. Mat. Astr. Fys. **1**, 327–345 (1904); Cartwright, Proc. London Math. Soc. (2) **33**, 209–224 (1931)] is developed. If the zeros have too large a defect, a contradiction with the results of the analysis of gap series arises. This is the essence of the argument which is however elaborated to give the following statement. If $F(z)=F_1(z)/F_2(z)=\sum_0^\infty a_n z^{\lambda_n}/\sum_0^\infty b_n z^{\mu_n}$ is a meromorphic function of integer order ρ and proximate order $\rho(r)$, then there exists at most one meromorphic function $f(z)$ of lower order (in relation to $\rho(r)$ and below) such that the zeros of $F(z)-f(z)$ are of lower order when the maximal density of $\{\lambda_n\}$ is less than $1/\rho$ and $F_2(z)$ is of lower order than $F_1(z)$, or if the maximal density of $\{\lambda_n\}$ is less than $1/2\rho$ if $F_1(z)$ and $F_2(z)$ are of the same order. (A modified theorem for exceptional values is stated.)

These results are also extended to functions of infinite order using the theory of Hiong [J. Math. Pures Appl. (9) **14**, 233–308 (1935)]. The gap conditions are now more drastic ($\lambda_n>n^{1+\delta}$) to make use of work by Mandelbrojt

[Trans. Amer. Math. Soc. 55, 96–131 (1944); these Rev. 5, 176] and a special definition of defect depending on the value of δ is required. *A. J. Macintyre* (Aberdeen).

Schaeffer, A. C., and Spencer, D. C. Models illustrating the third coefficient region for schlicht functions. Scripta Math. 16, 67–71 (2 plates) (1950).

Grunsky, Helmut. Nachtrag zu meinen Arbeiten über "Eindeutige beschränkte Funktionen in mehrfach zusammenhängenden Gebieten." Math. Z. 52, 852 (1950).

This is a complement to parts I and II of the paper cited in the title [Jber. Deutsch. Math. Verein. 50, 230–255 (1940); 52, 118–132 (1949); these Rev. 2, 275; 4, 270]. The uniqueness theorems that were originally lacking are proved briefly. Part III [Math. Z. 51, 586–615 (1949); these Rev. 10, 441] is thereby supplanted. *M. Heins.*

Baganas, Nicolas. Un théorème général sur les fonctions algébroïdes. C. R. Acad. Sci. Paris 230, 1728–1730 (1950).

Statement without proof of a theorem concerning algebroid functions for which the defining equation has coefficients which are themselves algebroid functions. Applications to the theory of entire functions are given. *M. Heins.*

Ahiezer, N. I. Concerning the approximation of continuous functions on the whole real axis. I. Zapiski Naučno-Issled. Inst. Mat. Meh. Har'kov. Mat. Obšč. (4) 19, 21–25 (1948). (Russian)

S. Bernstein [C. R. (Doklady) Acad. Sci. URSS (N.S.) 51, 331–334, 487–490 (1946); 52, 563–566 (1946); 54, 103–108, 475–478 (1946); these Rev. 8, 20, 323, 373, 509] determined the best approximation on $(-\infty, \infty)$ to $(\alpha+\beta x)/(x^2+c^2)$ by entire functions of exponential type T, and the asymptotic value of the best approximation to $(\alpha+\beta x)/(x^2+c^2)^n$ as $T\to\infty$. The author proves these results in another way, by showing that when $\varphi(x)$ is an entire function of exponential type T and a_k, b_k are real, then the least upper bound on $(-\infty, \infty)$ of the absolute value of

$$\Delta \equiv \frac{A}{(x-ic)^n}+\frac{\bar{A}}{(x+ic)^n}-\sum_{k=1}^{n-1}\frac{a_k+b_kx}{(x^2+c^2)^k}-\varphi(x)$$

is at least $L=2^{-n+1}|A|c^{-n}e^{-Tc}$. There is equality only when Δ is a multiple of

$$y\equiv e^{-iTx-i\beta}\left(\frac{x+ic}{x-ic}\right)^n+e^{iTx+i\beta}\left(\frac{x-ic}{x+ic}\right)^n.$$

For the proof note that the maximum of $|y|$ is L; let z be another expression of the form Δ, with ψ instead of φ, and let $M=\sup|z|$. By applying the maximum principle to

$$h(x)=e^{iTx}(x-ic)^n(x+ic)^{-n}M^{-1}\{z+(z^2-M^2)^{\frac{1}{2}}\}$$

with an appropriate branch of the radical, in an appropriate domain of the x-plane, one shows that $|h(x)|\leqq 1$, hence $h(x)\equiv e^{\Omega}$, and by letting $x=ic$ the result follows.

R. P. Boas, Jr. (Evanston, Ill.).

Macintyre, A. J., and Rogosinski, W. W. Extremum problems in the theory of analytic functions. Acta Math. 82, 275–325 (1950).

The authors present a systematic theory of the following extremal problem: maximize $|I(f)|=|(2\pi i)^{-1}\int f(\zeta)k(\zeta)d\zeta|$, where integration is over the unit circle, f belongs to H_p, and k is given and belongs to the conjugate class consisting of functions regular in $|z|<1$ except for a finite number of poles, with $\int_0^{2\pi}|f(re^{i\theta})|^p d\theta$ bounded for r near 1 $(p^{-1}+q^{-1}=1)$. The problem is shown to be equivalent to an interpolation problem in H_q and the solution is shown to exist uniquely. The first part of the paper gives the general theory; the second part gives numerous applications, both to old problems formerly treated by special devices and to new problems. The following are a few of the special problems which are solved: maximize $\int_0^1|f(x)|dx$ in H_2, $\int_0^r|f(x)|dx$ in H; maximize $|f^{(n)}(\beta)|$ in H_p (solved for certain values of p and n); maximize the absolute value of the partial sums of the power series of $f(z)$ $(H_p,\ p\geqq 2)$. The authors also discuss the more general problem of the region of values for $I(f)$ when $\int_0^{2\pi}|f(e^{i\theta})|^p d\theta$ and the values of f at a finite number of points are prescribed.

R. P. Boas, Jr. (Evanston, Ill.).

Lehto, Olli. A method of analytic continuation. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 70, 11 pp. (1950).

In this paper, the author states in a more general form a result on analytic continuation already used by him in an earlier paper [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 59 (1949); these Rev. 11, 170]. If D is a domain in the complex z-plane bounded by the simple closed analytic curves $C_1, \cdots, C_n$ $(C_1+\cdots+C_n=C)$ and if $F(z)$ is single-valued and meromorphic in D and satisfies $\int_C|F(z)dz|<\infty$, the above result provides a necessary and sufficient condition for $F(z)$ to be regular on C. This condition is easily verified in the case of the solutions of certain extremal problems, which are thus shown to be regular on C. As already shown by the author in the previous paper cited above, this remark leads to new existence proofs for the parallel slit mappings and related domain functions.

Z. Nehari (St. Louis, Mo.).

Garabedian, P. R., and Schiffer, M. On existence theorems of potential theory and conformal mapping. Ann. of Math. (2) 52, 164–187 (1950).

In this paper, the authors give new proofs for some of the existence theorems of the theory of conformal mapping and of potential theory. The proofs are based on the concept of the kernel function of a complete set S of functions with respect to a metric M introduced by Bergman [Mémor. Sci. Math. nos. 106, 108, Gauthier-Villars, Paris, 1947, 1948; these Rev. 11, 344]. The basic idea of these proofs is roughly the following. The kernel function of S is set up and the inner product P, corresponding to the metric M, of the kernel function and the function representing the fundamental singularity associated with the set S is formed. A detailed study of the behavior of P if the fundamental singularity crosses the boundary of the domain of definition of the set S then yields information regarding the boundary behavior of the kernel function which is substantially equivalent to the desired existence proofs. This procedure is carried through in four cases, one of which, involving the existence of the parallel slit mappings, has recently also been disposed of by Lehto [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 59 (1949); these Rev. 11, 170] by a method which is essentially the same as that of this paper. Generalizations are indicated.

Z. Nehari (St. Louis, Mo.).

***Courant, R. Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces. Appendix by M. Schiffer.** Interscience Publishers, Inc., New York, N. Y., 1950. xiii+330 pp. $4.50.

In this book the author presents the modern development of the Dirichlet principle in a manner which is simple, yet independent of the advanced notions of Hilbert space. Physical interpretations and motivations are stressed throughout, and proofs are given with clarity and detail, yet with sufficient assumption of an elementary knowledge to attract the reader who is chiefly interested in research in this field. In chapter I, the fundamental boundary value problem of potential theory is solved by means of two alternative proofs of Dirichlet's principle. Chapter II gives applications to conformal mappings on canonical slit domains, and, in addition to particular cases, the uniformization of an arbitrary abstract Riemann surface is treated. Chapter III is devoted to the solution of the problem of Plateau for simply-connected surfaces by means of the author's original attack through the Dirichlet principle. In chapter IV the Plateau problem is treated for surfaces of higher connectivity and genus, and in chapter V the results are applied to the two-dimensional case in order to obtain the existence of canonical conformal mappings on regions bounded by circles, or, more generally, by convex curves. Chapter VI concludes the main part of the work with a more advanced discussion of free boundary problems and unstable minimal surfaces. One quarter of the book is devoted to an appendix by Schiffer on recent developments in conformal mapping. Here the leading problems are drawn from the theory of the various kernel functions and their relation to the Green's function, with the Dirichlet integral providing a principal tool for manipulation. Hadamard's variational formula for Green's function and the related theory of interior variations are presented and are illustrated with examples in the study of univalent functions. The appendix serves to emphasize the close connection between the investigations centering about the kernel function and the more classical results which have grown out of Dirichlet's principle. Altogether, the book offers a presentation in unified form of the material now available in the various related theories of the Dirichlet problem, conformal mapping of Riemann surfaces, and minimal surfaces. *P. R. Garabedian.*

Ahlfors, Lars V. Open Riemann surfaces and extremal problems on compact subregions. Comment. Math. Helv. 24, 100–134 (1950).

Making use of the exterior calculus of differentials, the author discusses conditions for an open Riemann surface W_0 to have, in various senses, a zero boundary. The results depend upon the solution in compact subregions W of W_0 of extremal problems treated by Grunsky [Jber. Deutsch. Math. Verein. 52, 118–132 (1942); these Rev. 4, 270] and Schiffer [Duke Math. J. 13, 529–540 (1946); these Rev. 8, 371] for multiply-connected plane regions. The extremal problems are solved in terms of Schottky differentials, that is, in terms of differentials which can be continued analytically onto the symmetric double of W. It is shown that if there are no nontrivial exact analytic differentials of finite norm in the sense of the Dirichlet integral on W_0, then every analytic differential Φ_0 of finite norm on W_0 is the limit of Schottky differentials φ defined on compact subregions W of W_0, with a similar result for harmonic differentials. It is also proved that every single-valued, bounded analytic function on W_0 reduces to a constant if and only if, given $\epsilon>0$, given a compact subset E of W_0, and given points a, b in W_0, there exists a compact subregion $W \subset W_0$ containing a, b, and E and a differential Φ in W with a simple pole of residue 1 at b and a simple pole at a such that $\int_C |\Phi| < \epsilon$, where C is the boundary of W. *P. R. Garabedian.*

Myrberg, Lauri. Normalintegrale auf zweiblättrigen Riemannschen Flächen mit reellen Verzweigungspunkten. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 71, 51 pp. (1950).

This paper is concerned with the theory of Abelian integrals of the first kind on two-sheeted Riemann surfaces with only real ramification points. Earlier investigations of this type [Hornich, Monatsh. Math. Phys. 47, 380–387 (1939); these Rev. 1, 49; P. J. Myrberg, Acta Math. 76, 185–224 (1944); same Ann. Ser. A. I. Math.-Phys. no. 31 (1945); these Rev. 7, 57, 428] were confined to the case where the ramification points clustered at only one point. In the present investigation no such restrictive assumptions are made and surfaces with positive boundary as well as those with null boundary are considered. Attention is given to special integrals of the first kind which are termed normal and are subject to conditions of reality and boundedness. Elementary normal integrals are introduced with the aid of which every normal integral is representable as a convergent infinite series. The behavior of normal integrals in the neighborhood of a cluster point of the ramification points is studied. Finally the dependence of normal integrals on their periods is studied for surfaces with positive boundary.
M. Heins (Providence, R. I.).

Bünding, Hilde. Riemannsche Flächen bei z^z und verwandten ganzen Transzendenten. Mitt. Math. Sem. Univ. Giessen 32, ii+34 pp. (1944).

Detailed discussion of the Riemann surfaces associated with the function z^z and with some allied functions.
Z. Nehari (St. Louis, Mo.).

Fischer, Wilhelm. Über die Riemann'sche Fläche der Gauss'schen Ψ-Funktion und der Mittag-Leffler'schen E_a-Funktionen. Mitt. Math. Sem. Univ. Giessen 37, i+35 pp. (1949).

Detailed discussion of the Riemann surfaces associated with the functions mentioned in the title. *Z. Nehari.*

Rutishauser, Heinz. Über Folgen und Scharen von analytischen und meromorphen Funktionen mehrerer Variabeln, sowie von analytischen Abbildungen. Acta Math. 83, 249–325 (1950).

[Also issued as a thesis by the Eidgenössische Technische Hochschule in Zürich, 1950.] Verf. überträgt die hauptsächlichen Ergebnisse von Montel über normale und quasinormale Scharen analytischer und meromorpher Funktionen einer komplexen Veränderlichen auf solche von Funktionen zweier Veränderlichen und führt so die Arbeiten von Julia und Saxer fort [P. Montel, Leçons sur les familles normales de fonctions analytiques et leur applications, Gauthier-Villars, Paris, 1927; G. Julia, Acta Math. 47, 53–115 (1925); W. Saxer, Comment. Math. Helv. 4, 256–267 (1932)]. Bei diesen Übertragungen ist grundlegend, dass man den isolierten Singularitäten der Funktionen einer Veränderlichen isolierte analytische Flächen und der Anzahl der a-Stellen die "Blätterzahl" einer analytischen Fläche im gegebenen Bereich entsprechen lässt. Eine im Bereiche $\mathfrak{B}$ analytische Fläche wird (etwas abweichend von der gebräuchlichen Definition) als Nullstellenmannigfaltigkeit einer in $\mathfrak{B}$ mero-

morphen Funktion aufgefasst. Es gilt der für das Weitere fundamentale Satz (Satz 2): Eine Schar von in einem Bereiche $\mathfrak{B}$ analytischen Flächen, deren Blätterzahlen oder Flächeninhalte in $\mathfrak{B}$ gleichmässig beschränkt sind, besitzt stets eine Teilfolge, die gegen eine in $\mathfrak{B}$ überall analytische Fläche konvergiert (der Fall einer Folge ohne Häufungspunkte in $\mathfrak{B}$ ist hier einbegriffen). Jedem Zweig Z der Grenzfläche V_0 einer konvergenten Folge analytischer Flächen V_j lässt sich eine "Belegungszahl" n_Z zuordnen (die mit den Blätterzahlen der Flächen V_j in einer genügend kleinen Schlauchumgebung von V_0 zusammenhängt).

Eine Folge θ in $\mathfrak{B}$ analytischer Funktionen heisst quasiregulär in $\mathfrak{B}$, falls ihre irregulären Punkte eine überall in $\mathfrak{B}$ analytische Fläche V_θ, die Irregularitätsfläche der Folge, bilden. Wie in der klassischen Theorie konvergiert eine quasireguläre Folge in jedem abgeschlossenen Teilgebiet von $\mathfrak{B}$, das keine Punkte von V_θ enthält, gleichmässig gegen die Konstante ∞ (Satz 7). Die Folge θ heisst irreduzibel in $\mathfrak{B}$, falls jede Teilfolge in jedem Punkte von V_θ irregulär ist. Wie in der klassischen Theorie wird eine Schar von im Bereiche $\mathfrak{B}$ analytischen Funktionen quasinormal genannt, falls jede Folge der Schar eine in $\mathfrak{B}$ reguläre oder quasireguläre Teilfolge besitzt. Existiert die obere Grenze s der Blätterzahlen der Irregularitätsflächen V_θ aller irreduzibeln Folgen einer Schar, so heisst s ihre Ordnung. Das bekannte Kriterium der zwei Quasiausnahmewerte (a, b) lässt sich sinngemäss übertragen (Satz 9): Wenn für alle Funktionen einer in $\mathfrak{B}$ gegebenen Schar eine gemeinsame obere Grenze p für die Blätterzahl der a-Stellenflächen und eine obere Grenze q für die der b-Stellenflächen in $\mathfrak{B}$ existieren, so ist die Schar quasinormal in $\mathfrak{B}$ von endlicher Ordnung s, wobei $s \leqq p, q$. Es lassen sich nun auch die "totale Ordnung" einer quasinormalen Schar einführen und die bekannten Bedingungen übertragen, nach denen eine quasinormale Schar total endlicher Ordnung normal ist, woraus dann Verallgemeinerungen der Sätze von Schottky und Landau gewonnen werden.

Analog werden (3tes Kapitel) quasireguläre Folgen und quasinormale Scharen meromorpher Funktionen untersucht. Das 4te Kapitel bringt Anwendungen auf ganze Funktionen (Kriterium, damit eine ganze Fläche algebraisch sei; die Julia-Geraden einer ganzen Funktion). Das 5te Kapitel behandelt Folgen analytischer und meromorpher Abbildungen allgemeinster Art, die durch in $\mathfrak{B}$ analytische oder meromorphe Funktionspaare gegeben sind. Es können hierbei isolierte irreguläre Stellen auftreten. Doch hat die Gesamtheit der irregulären Punkte die gleichen Eigenschaften wie die der irregulären Punkte einer Folge meromorpher Funktionen (Satz 28). Das Bild einer isolierten irregulären Stelle einer Folge analytischer Abbildungen ist der ganze abgeschlossene Raum (Satz 29). Es werden sinngemäss quasireguläre Abbildungsfolgen, normale und quasinormale Scharen meromorpher Abbildungen und quasinormale Scharen total endlicher Ordnung definiert (Kapitel 6) und ähnliche Eigenschaften wie für Funktionsfolgen und Scharen nachgewiesen, unter anderem Verallgemeinerungen des Satzes von Landau (Satz von Bureau) und des Schottky'schen Satzes.

[Zur vorstehenden Arbeit siehe auch Rutishauser, C. R. Acad. Sci. Paris **224**, 1804–1806 (1947); **225**, 33–35 (1947); diese Rev. **9**, 25.] *P. Thullen* (Panamá).

Sommer, Friedrich. Die Geometrie der Hyperkugelautomorphismen. Schr. Math. Inst. Univ. Münster, no. 3, iii+143 pp. (1949).

This is a far-reaching classification, in matrix language, of the holomorphic (fractional linear) isomorphisms of the unit sphere K^{2k}: $z_1\bar{z}_1+\cdots+z_k\bar{z}_k<1$, which will be a useful prerequisite for analyzing the Fuchsian groups in it, such as may exist. Among other things, the author shows that, topologically (not group-isomorphically), the group space of such isomorphisms is the product of K^{2k} with the group space of unitary transformations D^{k^2}. *S. Bochner.*

Esteban Carrasco, Luis. The nth rectilinear derivative of a polygenic function. Gaceta Mat. (1) **1**, 11–25 (1949). (Spanish)

The first derivative $\gamma=dw/dz$ of a polygenic function $w=\phi(x, y)+i\psi(x, y)$ was originally studied by Kasner [Science (N.S.) 66, 581–582 (1927)] who proved that the derivative is represented by a congruence of clocks. In a paper by Kasner and De Cicco [Univ. Nac. Tucumán. Revista A. 4, 7–45 (1944); these Rev. 7, 59] the second derivative $\sigma=d^2w/dz^2$ was studied along any curvilinear path. The related geometry established that σ is represented by a congruence of limaçons, which includes a simple family of cardioids. The author of this paper studies the rectilinear nth derivative $\lambda=d^nw/dz^n$ of a polygenic function w. At a fixed point z, this nth derivative is represented by the equation: $\lambda=a_0+a_1t+\cdots+a_nt^n$, where $a_0, \cdots, a_n$ are complex constants and t is a complex parameter varying over the unit circle with center at the origin. The locus of λ is a closed curve of Pascal of the $(n-1)$th kind. If $n=1$, the Kasner derivative circle is obtained. If $n=2$, the locus is a limaçon. These Pascal curves of the $(n-1)$th kind are algebraic curves of degree $2n$. The case $n=3$ is studied in complete detail. *J. De Cicco* (Chicago, Ill.).

Theory of Series

Levitan, B. M. Estimation of the remainder term in the formula of Taylor-Delsarte. Doklady Akad. Nauk SSSR (N.S.) **73**, 269–272 (1950). (Russian)

Let $Lv(x)=v''-g(x)v, x\geqq 0$, and let $\sum_{k=0}^{n}\lambda^k\phi_k(x)$ be the solution of the differential equation $Lv=\lambda v$, which satisfies the initial conditions $v(0)=1, v'(0)=0$. If $T(x, y; f)$ is the solution of the partial differential equation $u_{xx}-g(x)u=u_{yy}-g(y)u$, which satisfies the initial conditions $u(x, 0)=f(x), u_y(x, 0)=0$, then the following theorem is proved for the case that $g(x)$ is continuous in every finite interval, nonnegative, and monotone nonincreasing:

$$T(x, y; f)=\sum_{k=0}^{n}\phi_k(y)L^{(k)}f(x)+R_n(x, y; f),$$

where $R_n(x, y; f)=\phi_{n+1}(y)L^{(n+1)}f(x+\theta y)$, $|\theta|<1$, provided $L^{(k)}f(x)$ exists for all integral k such that $1\leqq k\leqq n+1$. This generalization of Taylor's theorem was suggested by J. Delsarte [J. Math. Pures Appl. (9) **17**, 213–231 (1938)]. An example is given to show that the conclusion in the theorem may hold even under less restrictive hypotheses on $g(x)$ than those made in the theorem. *H. P. Thielman.*

Cowling, V. F. Summability and analytic continuation. Proc. Amer. Math. Soc. **1**, 536–542 (1950).

Power-series methods of summability, which depend upon a complex parameter r and involve the transformation

$$t_n=\sum_{m=n}^{\infty}(1-r)^{n+1}\binom{m}{n}r^{m-n}s_m,$$

are studied and applied to power series. As is pointed out

in a note added in proof, similar results have been obtained by others; see W. Meyer-König [Math. Z. 52, 257–304 (1949); these Rev. 11, 242] and other references given by Meyer-König and the reviewer. *R. P. Agnew.*

Erdös, P., and Piranian, G. Convergence fields of row-finite and row-infinite Toeplitz transformations. Proc. Amer. Math. Soc. 1, 397–401 (1950).

It is known that in the space of bounded sequences, every regular Toeplitz transformation is equivalent to some row-finite regular transformation [Banach, Théorie des Opérations Linéaires, Warsaw, 1932, p. 72]. The authors show that the corresponding result is false when unbounded sequences are considered. They construct a regular Toeplitz transformation which has a convergence field not contained in that of any regular row-finite Toeplitz transformation. Each row of the matrix of this transformation contains infinitely many nonzero elements and each column precisely one nonzero element. It is also shown that a Hölder transformation is equivalent to a matrix transformation with positive elements and that the convergence field of a row-finite transformation is not necessarily contained in that of any row-infinite transformation. *H. G. Eggleston.*

Morley, Hilda. A theorem on Hausdorff transformations and its application to Cesàro and Hölder means. J. London Math. Soc. 25, 168–173 (1950).

The result, that each conservative Hausdorff method H is also absolutely conservative, and its applications to Cesàro and Hölder methods were given earlier by Knopp and Lorentz [Arch. Math. 2, 10–16 (1949); these Rev. 11, 346] to whom the author refers in a note added in proof.

R. P. Agnew (Ithaca, N. Y.).

Žak, I. E. Absolute summability of double numerical series. Doklady Akad. Nauk SSSR (N.S.) 73, 639–642 (1950). (Russian)

A double series $\sum a_{mn}$ is called absolutely summable by the Cesàro method $|C(\alpha, \beta)|$ if its $C(\alpha, \beta)$ transform belongs to the class H of sequences $u_{m,n}$ for which

$$\sum |u_{m,n}-u_{m-1,n}-u_{m,n-1}-u_{m-1,n-1}|<\infty,$$

$$\sum |u_{m,n}-u_{m-1,n}|<\infty, \quad \sum |u_{m,n}-u_{m,n-1}|<\infty.$$

If $\alpha, \beta>-1$ and $h, k>0$, then each series $\sum a_{mn}$ summable $|C(\alpha, \beta)|$ is also summable $|C'(\alpha+h, \beta+k)|$. Conditions on the coefficients of a trigonometric series are found which imply that the series is summable $|C(\alpha, \beta)|$ almost everywhere whenever $\alpha, \beta>\frac{1}{2}$. [The author's scrambled reference "T. Wang, Duke Math. Journ., 9, 803 (1942)" should be to the first of the two papers: F. T. Wang, Duke Math. J. 9, 567–572 (1942); M. T. Cheng, ibid. 9, 803–810 (1942); these Rev. 4, 37, 156. These treat, respectively, absolute summability and absolute convergence of simple Fourier series.]

R. P. Agnew (Ithaca, N. Y.).

Popovitch, B. La liaison des procédés de sommabillité avec les intervalles de convergence. Bull. Soc. Math. Phys. Serbie 1, no. 3–4, 121–130 (1949). (Serbian. French summary)

For each of 17 different functions $\Lambda(t)$, the author tabulates a function $T_\lambda(t)$ such that $\lim_{x\to\infty}(1/\Lambda(x))\int_0^x s(t)d\Lambda(t)=s$ implies $\lim s(t)=s$ whenever $s(t)$ satisfies a Tauberian condition (of E. Schmidt slowly-decreasing type) $\liminf_{t,t'\to\infty}\{s(t')-s(t)\}\geqq -w(\lambda)$, where t' is subject to the restriction $t\leqq t'\leqq T_\lambda(t)$ and $w(\lambda)\to 0$ as $\lambda\to 1-$. The functions $\Lambda(t)$ considered involve logarithms, exponentials, iterated logarithms, and iterated exponentials in various ways. The functions $T_\lambda(t)$ are calculated from the known formula $T=V(\lambda\Lambda(t))$, V being the inverse function of Λ. [For an exposition of the subject, see Karamata, Sur les théorèmes inverses des procédés de sommabilité, Actual. Sci. Ind., no. 450, Hermann, Paris, 1937.] *R. P. Agnew.*

Kalašnikov, M. D. A remark on infinite products. Doklady Akad. Nauk SSSR (N.S.) 73, 9–12 (1950). (Russian)

A real sequence $\{a_k\}$ may be such that if $\prod(1+u_k)$ is any convergent real infinite product, $\prod(1+a_ku_k)$ also converges. For the a_k to have this property, the following two conditions are shown to be necessary and sufficient. First, the a_k must tend either towards zero or towards unity as k increases. Secondly, a being the limit of the a_k, $\sum(a_k-a)$ must converge absolutely. *J. F. Ritt* (New York, N. Y.).

Fourier Series and Generalizations, Integral Transforms

Schoenberg, I. J. The finite Fourier series and elementary geometry. Amer. Math. Monthly 57, 390–404 (1950).

The coefficients (ξ_ν) of the uniquely defined polynomial $P(x)=\xi_0+\xi_1x+\cdots+\xi_{k-1}x^{k-1}$ satisfying the equations $P(e^{2\pi i\nu/k})=z_\nu$, $\nu=0, 1, \cdots, k-1$, are called the finite Fourier coefficients of the sequence (z_ν). After some geometric interpretations, the analogue of the Parseval relation

$$\sum_{\nu=0}^{k-1}|z_\nu|^2=k\sum_{\nu=0}^{k-1}|\xi_\nu|^2$$

and a theorem concerning the finite Fourier coefficients of a sequence generated from (z_ν) by a linear cyclic transformation are proved. Various geometric applications are made from which the following corollary of a general result should be mentioned. If O is the centroid of the points P_1, P_2, P_3, P_4, then $2\leqq(\overline{P_1P_2}^2+\cdots+\overline{P_4P_1}^2)/(\overline{OP_1}^2+\cdots+\overline{OP_4}^2)\leqq 4$ with equality on the left if and only if $P_1P_2P_3P_4$ is a parallelogram, and equality on the right if and only if $P_1=P_3$, $P_2=P_4$.

L. Fejes Tóth (Veszprém).

Kozlov, V. Ya. On the completeness of a system of functions of type $\{\varphi(nx)\}$ in the space L_2. Doklady Akad. Nauk SSSR (N.S.) 73, 441–444 (1950). (Russian)

Let $A(f_1, f_2, \cdots)$ denote the set of functions

$$\{f_1(nt), f_2(nt), \cdots\},$$

$n=1, 2, 3, \cdots$. Theorem 1. If

$$\varphi_j(t)\sim\sum_{n=1}^{\infty}a_{jn}\cos nt+\sum_{n=1}^{\infty}b_{jn}\sin nt, \qquad j=1, 2,$$

are two functions of $L^2(0, 2\pi)$ such that $A(1, \varphi_1, \varphi_2)$ is complete in $L^2(0, 2\pi)$, then the sets $A(1, \psi_1, \psi_2)$ and $A(1, \chi_1, \chi_2)$ are also complete in $L^2(0, 2\pi)$, where

$$\psi_j\sim\sum a_{jn}f(n)\cos nt+\sum b_{jn}f(n)\sin nt,$$
$$\chi_j\sim\sum b_{jn}f(n)\cos nt-\sum a_{jn}f(n)\sin nt,$$

and $f(n)$ is a multiplicative function $[f(k)f(l)=f(kl)]$ with $|f(n)|\leqq 1$. Corollary. Under the hypothesis of the theorem $A(1, \int_0^t\varphi_1(u)du, \int_0^t\varphi_2(u)du)$ is complete in $L^2(0, 2\pi)$. Theorem 2. If φ_1 and φ_2 satisfy the conditions of theorem 1, and if $g_j=\sum_{n=1}^{N}A_{jn}\varphi_1(nt)+\sum_{n=1}^{N}B_{jn}\varphi_2(nt)$, $j=1, 2$, then

$A(1, g_1, g_2)$ is complete in $L^2(0, 2\pi)$ if and only if

$$\begin{vmatrix} \sum_{n=1}^{N} A_{1n}f(n) & \sum_{n=1}^{N} B_{1n}f(n) \\ \sum_{n=1}^{N} A_{2n}f(n) & \sum_{n=1}^{N} B_{2n}f(n) \end{vmatrix} \neq 0$$

for every multiplicative function f with $|f(n)|<1$. The theorems are stated in a slightly more general Hilbert-space formulation. The problem of the completeness of $\{\varphi(nx)\}$ in the space of odd functions of period 2π in $L^2(0, 2\pi)$ is raised, where

$$\varphi(x)=1 \ (0<x<\alpha), \quad \varphi(x)=0 \ (\alpha \leqq x \leqq \pi),$$
$$\varphi(x)=-\varphi(-x)=\varphi(x+2\pi).$$

The set is complete for $\alpha=\pi, \pi/2, 2\pi/3$, but is not complete for α in a certain neighborhood of $\pi/3$ and for $\alpha=q\pi/p$ (p an odd prime, q odd, $\tan^2(q\pi/2p)<1/p$).

W. H. J. Fuchs (Ithaca, N. Y.).

Bicadze, A. V. On a system of functions. Uspehi Matem. Nauk (N.S.) **5**, no. 4(38), 154–155 (1950). (Russian)

Theorem: A function $f(\theta)$ satisfying a Hölder condition on $[0, \pi]$ has a uniformly convergent expansion

$$\sum_{0}^{\infty} a_n(\cos n\theta+\sin n\theta).$$

The proof is based on the construction of a regular function $F(z)=u+iv$ in $|z|<1$ such that $u \to f(\theta)$ on the upper half of $|z|=1$ and $v \to -f(-\theta)$ on the lower half.

R. P. Boas, Jr. (Evanston, Ill.).

Nikol'skiĭ, S. On the Dini-Lipschitz condition for convergence of Fourier series. Doklady Akad. Nauk SSSR (N.S.) **73**, 457–460 (1950). (Russian)

The author rediscovers the result of Lebesgue [Bull. Soc. Math. France **38**, 184–210 (1910)] to the effect that the Dini-Lipschitz condition for the convergence of Fourier series cannot be improved [see also the reviewer, Trigonometrical Series, Warszawa-Lwów, 1935, p. 171].

A. Zygmund (Chicago, Ill.).

Žak, I. E. On some properties of conjugate double trigonometric series. Doklady Akad. Nauk SSSR (N.S.) **73**, 5–8 (1950). (Russian)

Let $f(x, y)$ be of period 2π in both x and y. It is said to be of class Lip (α, β) if

$$|f(x_1, y_1)-f(x_2, y_1)+f(x_1, y_2)-f(x_2, y_2)| \leqq M|x_1-x_2|^{\alpha}|y_1-y_2|^{\beta},$$

where M is independent of x_1, y_1, x_2, y_2. It is shown that, if $f \in$ Lip (α, β), where $0<\alpha<1$, $0<\beta<1$, then the series conjugate to the Fourier series of f converges to a function $\bar{f}(x, y) \in$ Lip (α, β). By the conjugate of a double trigonometric series the author means the formal conjugate with respect to both x and y. [See also L. Cesàri, Ann. Scuola Norm. Super. Pisa (2) **7**, 279–295 (1938).]

A. Zygmund (Chicago, Ill.).

Morozov, M. I. Approximation of functions satisfying a Lipschitz condition by means of interpolation polynomials with double points of interpolation. Uspehi Matem. Nauk (N.S.) **5**, no. 4(38), 156–161 (1950). (Russian)

The author determines the upper bound for certain classes of $f(x)$ of the maximum of the remainder in Hermite interpolation for two choices of the interpolation points: trigonometric interpolation to a function of period 2π at the points $x_0+2k\pi/n$ (Jackson polynomials) and polynomial interpolation on $(-1, 1)$ with $f(x)$ and $f'(x)$ assigned at the points $\cos(k-\frac{1}{2})\pi/n$. In the first case $f(x)$ satisfies

$$|f(x')-f(x'')|<|x'-x''|^{\alpha},$$

$0<\alpha \leqq 1$, and the upper bound of the remainder is

$$4\pi^{-\alpha} \max \{\sin^2 \tfrac{1}{2}u \cdot \sum_{-\infty}^{\infty} |u-2k\pi|^{2-\alpha}\}+o(n^{-\alpha})$$

for $\alpha<1$, and $4\pi^{-1}n^{-1}\log n+O(n^{-1})$ for $\alpha=1$. The second case is considered only for $\alpha=1$, and the same bound is given, but not shown to be attained except for odd n.

R. P. Boas, Jr. (Evanston, Ill.).

Remez, E. Ya. Detailed investigations of limiting relations between power-mean and Čebyšev approximations. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. **1948**, no. 10, 107–141 (1948). (Ukrainian. Russian summary)

Let $v_0(x), v_1(x), \cdots, v_{n+1}(x)$ be $n+1$ given continuous functions in the interval $0 \leqq x \leqq 1$, and let $c_1, c_2, \cdots, c_n$ be any numbers. Let

$$\Phi(x)=v_0(x)+\sum_{1}^{n} c_i v_i(x),$$

$$\delta_m[\Phi]=\delta_m(c_1, \cdots, c_n)=\left(\int_0^1 |\Phi(x)|^m dx\right)^{1/m},$$

$$\delta_0[\Phi]=\sup_{0 \leqq x \leqq 1} |\Phi(x)|$$

for $m>1$. The problem of the existence and the uniqueness of the numbers c_i minimizing $\delta_m[\Phi]$ was first discussed by Pólya [C. R. Acad. Sci. Paris **157**, 840–843 (1913)] and Jackson [Trans. Amer. Math. Soc. **22**, 117–128 (1921)] who also proved that $\delta_0[\Phi_m] \to \delta_0[\Phi_0]$ for $m \to \infty$, where Φ_m is the function minimizing δ_m. Thus $\delta_0[\Phi_m]=(1+\alpha_m)\delta_0[\Phi_0]$ with α_m tending to 0 as $m \to \infty$. It is the study of the rapidity with which α_m tends to zero which is the main topic of the paper. It is shown that if all the v_j satisfy Lipschitz conditions of positive order, then $\alpha_m=O(m^{-1}\log m)$. More precisely, if $v_j \in$ Lip τ, $\tau>0$, for all j, then

$$\limsup (m\alpha_m/\log m) \leqq 1/\tau.$$

If all the v_j have modulus of continuity $O(\log \delta^{-1})^{-1}$, then $\alpha_m=O(m^{-1})$. Examples show that these estimates are best possible. Corresponding, though less simple, theorems exist for general moduli of continuity. For general continuous v_j the numbers α_m may tend to 0 arbitrarily slowly, at least for some values of m.

A. Zygmund (Chicago, Ill.).

Zamansky, Marc. Sur l'approximation des fonctions absolument continues à variation bornée. C. R. Acad. Sci. Paris **230**, 1730–1732 (1950).

Let $\varphi(x)$ be L-integrable of period 2π with mean value zero, and let $f(x)$ be an integral of $\varphi(x)$. Let

$$\Delta_2 f(x, u)=f(x+u)-2f(x)+f(x-u).$$

Then for the $(C, 1)$ means of $\sigma_n(x)$ of the Fourier series of f, we have the following relations

$$\sigma_n(x)-f(x)=(2n\pi)^{-1}\int_{1/n}^{\pi} \Delta_2 f(x, 2t)(\sin t)^{-2}dt+o'(1/n),$$

$$\bar{f}(x)-\bar{\sigma}_n(x)=n^{-1}\varphi(x)+o'(1/n),$$

where the prime indicates that the relation is valid at almost every point x. Extensions to the case of more general methods of summability follow.

A. Zygmund.

Zamansky, Marc. Sur les séries de Fourier. C. R. Acad. Sci. Paris **230**, 2256–2258 (1950).

The main theorem of the note asserts that if $f(x)$ is absolutely continuous, an integral of $\varphi(x)$, and of period 2π, then the conjugate function $\tilde{f}(x)$ has the following property: there is a set E whose complement is of measure zero and such that in E, and with respect to E, the function $\tilde{f}(x)$ is differentiable and has $\tilde{\varphi}(x)$ for derivative. In this form the result is not correct. That almost everywhere $\tilde{f}(x)$ has $\tilde{\varphi}(x)$ as approximate derivative has been proved by Titchmarsh [Proc. London Math. Soc. (2) **29**, 49–80 (1928)].

A. Zygmund (Chicago, Ill.).

Natanson, I. P. On the accuracy of representation of continuous periodic functions by singular integrals. Doklady Akad. Nauk SSSR (N.S.) **73**, 273–276 (1950). (Russian)

Let $f(t)$ be of period 2π and continuous. Let

$$f_n(x)=\int_{-\pi}^{\pi} f(t)\Phi_n(t-x)dt,$$

where $\Phi_n(t)$ is of period 2π, positive, even, and such that $\int_{-\pi}^{\pi}\Phi_n(t)dt=1$, and $\delta_n=\int_0^{\pi}t\Phi_n(t)dt$ tends to 0 as $n\to\infty$ (this condition is satisfied by the kernels of Poisson, Fejér, Jackson, and de La Vallée-Poussin). The author shows that if $f(t)$ has modulus of continuity $\omega(\delta)$, then $|f_n(x)-f(x)|\leqq 3\omega(\delta_n)$. The assumptions concerning Φ_n imply that the sequence $\Delta_n=\int_0^{\pi}t^2\Phi_n(t)dt$ tends to 0. Let us assume that for every fixed σ, $0<\sigma<\pi$, we also have $\int_{\sigma}^{\pi}\Phi_n(t)dt=o(\Delta_n)$. Then, for every $f(t)$ continuous at $t=x$, there is a finite number $f^*(x)$ such that $f_n(x)=f(x)+f^*(x)\Delta_n+o(\Delta_n)$. *A. Zygmund.*

Safronova, G. P. On a method of summation of divergent series related to Jackson's singular integral. Doklady Akad. Nauk SSSR (N.S.) **73**, 277–278 (1950). (Russian)

The Jackson integral

$$J_n(x)=\frac{3}{2n\pi(2n^2+1)}\int_{-\pi}^{\pi} f(t)\left\{\frac{\sin n\frac{1}{2}(t-x)}{\sin\frac{1}{2}(t-x)}\right\}^4 dt$$

can be written in the form

$$J_n(x)=\rho_0^{(n)}\tfrac{1}{2}a_0+\sum_{k=1}^{2n-2}\rho_k^{(n)}(a_k\cos kx+b_k\sin kx),$$

where a_k, b_k are the Fourier coefficients of f, and $\rho_k^{(n)}$ are numbers independent of f. Thus it leads to a method of summation of the Fourier series of f. The author proves that the method is not weaker than $(C, 3)$, and that in this assertion $(C, 3)$ cannot be replaced by $(C, 4)$.

A. Zygmund (Chicago, Ill.).

Pipes, Louis A. The summation of Fourier series by operational methods. J. Appl. Phys. **21**, 298–301 (1950).

A formal application of the Laplace transformation to certain trigonometric series leads to series of transforms that can be recognized as representing certain hyperbolic functions. When these functions are expanded in series of exponential functions the inverse transformation may lead to closed forms of the sums of the trigonometric series. This device of summing a series by recognizing the sum of the transformed series is illustrated here by four examples. As is pointed out in textbooks the use of Laplace transforms in solving some types of problems frequently points the way to a choice of forms of solutions, including closed forms. This paper uses that principle but it supposes that neither the function nor the transform of the function represented by the given Fourier series is known at the start.

R. V. Churchill (Ann Arbor, Mich.).

James, R. D. A generalized integral. II. Canadian J. Math. **2**, 297–306 (1950).

For the definition of the generalized integral and for references to other literature see the review of part I, written jointly with W. H. Gage [Trans. Roy. Soc. Canada. Sect. III. (3) **40**, 25–35 (1946); these Rev. **9**, 19] (the definition given in that review is slightly different from that given in part I, but is adopted by the author in part II). The main result of the present paper is that the sum of an everywhere convergent trigonometric series is integrable in the proposed sense, and the series itself is the Fourier series of the sum. The Fourier character of the series holds if we only assume that the coefficients of the series tend to zero and, except for a denumerable set of points, the upper and the lower Riemann sums of the series are finite. *A. Zygmund.*

Ghizzetti, Aldo. Sui coefficienti di Fourier di una funzione limitata, compresa fra limiti assegnati. Ann. Scuola Norm. Super. Pisa (3) **4**, 131–156 (1950).

Let $f(x)$ be a real integrable function of period 2π, c_k its complex Fourier coefficients. With $f(x)$ the author associates the function

$$F(x)=\exp\{-2\pi i\sum_{k=1}^{\infty}c_k z^k\}=1-ie^{ix}\sum_{k=1}^{\infty}s_k z^k,$$

defines $s_0=2\sin\pi c_0$, $s_{-k}=\bar{s}_k$, and introduces the determinants

$$D_k=\begin{vmatrix} s_0 & s_1 & \cdots & s_k \\ s_{-1} & s_0 & \cdots & s_{k-1} \\ \cdots & \cdots & \cdots & \\ s_{-k} & & \cdots & s_0 \end{vmatrix}.$$

He establishes a necessary and sufficient condition that $0\leqq f(x)\leqq 1$ almost everywhere: essentially the positivity of the D_k. His method involves the systematic use of step functions which take alternately the values 0 and 1 in a set of intervals filling $(0, 2\pi)$; he begins by establishing conditions for such a function to have assigned Fourier coefficients, and proceeds by means of an approximation process. The principal novelty of the paper is the method, which is more direct than methods which have been used in earlier work (quoted by the author), mostly dealing with functions having only a finite number of assigned coefficients.

R. P. Boas, Jr. (Evanston, Ill.).

Hirschman, I. I., Jr., and Jenkins, J. A. Note on a result of Levine and Lifschitz. Proc. Amer. Math. Soc. **1**, 390–393 (1950).

Lévine and Lifschetz [Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 693–711 (1941); these Rev. **3**, 106] considered the relationship between the gaps of a Fourier series and its admissible integral orders of zeros and have given a generalization of a corresponding theorem of Mandelbrojt [Séries de Fourier et classes quasi-analytiques de fonctions, Gauthier-Villars, Paris, 1935]. The authors improve the constants involved in the Lévine-Lifschetz theorems.

S. Mandelbrojt (Houston, Tex.).

Kawata, Tatsuo. Representation of a function by the Fourier-Stieltjes integral. Ann. Inst. Statist. Math., Tokyo **1**, 131–139 (1950).

Variants of Cramér's necessary and sufficient conditions for a function to be a Fourier-Stieltjes transform.

M. Loève (Berkeley, Calif.).

Levitan, B. M. Representation by Bessel functions in a Fourier-Stieltjes integral. Doklady Akad. Nauk SSSR (N.S.) **73**, 453–456 (1950). (Russian)

Let $j_p(\lambda^{\frac{1}{2}}x)=2^p\Gamma(p+1)x^{-p}\lambda^{-p/2}J_p(\lambda^{\frac{1}{2}}x)$, $p\geqq-\frac{1}{2}$,

$$T_p^x f(x)=\frac{\Gamma(p+1)}{\Gamma(\frac{1}{2})\Gamma(p+\frac{1}{2})}\int_0^\pi f[(x^2+y^2-2xy\cos\varphi)^{\frac{1}{2}}]\sin^{2p}\varphi\,d\varphi.$$

The author shows that an even continuous $f(x)$ can be represented in the form

$$\int_{-\infty}^{\infty} j_p(\lambda^{\frac{1}{2}}x)d\sigma(\lambda)$$

with σ nondecreasing and bounded, if and only if

$$\sum_{\mu,\nu=1}^{n} T_{x_\mu}^{x_\nu} f(x)\xi_\mu\xi_\nu\geqq 0$$

for arbitrary x_μ and ξ_μ and all n. *R. P. Boas, Jr.*

Bose, S. K. Some properties of the Whittaker transform. Bull. Calcutta Math. Soc. **42**, 43–48 (1950).

This is a sequel to the author's earlier papers [same Bull. **41**, 9–27, 59–67, 68–76, 221–222 (1949); these Rev. **11**, 28, 173, 174]. The author gives six theorems each of which is obtained by taking a recurrence formula, series, or other linear relation containing Whittaker functions, multiplying by an arbitrary function, and integrating. Each theorem is accompanied by examples. *A. Erdélyi.*

Sbrana, Francesco. Sopra alcuni problemi di propagazione in più dimensioni. Ann. Mat. Pura Appl. (4) **30**, 191–200 (1949).

[Cf. also the author, Atti Accad. Ligure **5**, 7–33, 173–186, 187–200, 201–217 (1949); Boll. Un. Mat. Ital. (3) **4**, 34–40 (1949); these Rev. **10**, 701, 702.] In this paper the author bases an operational calculus in two variables on the formula

$$f(x,y)=\lim_{\epsilon\to+0}(2\pi)^{-1}\iint_\sigma f(\xi,\eta)d\xi d\eta\int_0^\infty re^{-\epsilon r}J_0(rR)dr,$$

where σ is a sufficiently regular domain, f is continuous in σ, $P(x, y)$ and $Q(\xi, \eta)$ are points of σ, and R is the distance PQ. This is supplemented by a second formula concerning integrals over a curve in the (x, y)-plane. Clearly in this operational calculus $\partial^2/\partial x^2+\partial^2/\partial y^2$ corresponds to $-r^2$, and this is used to provide the explicit solution for the initial value problem of the partial differential equation $u_{xx}+u_{yy}-u_{tt}=0$ and for the boundary value problems of $u_{xx}+u_{yy}\pm k^2u=0$ (also for $k=0$). There is a corresponding result for three dimensional space with applications to the three dimensional wave equation and to elliptic equations in three variables. *A. Erdélyi* (Pasadena, Calif.).

Ditkin, V. A. A differential operator and the functional transformations connected with it. Uspehi Matem. Nauk (N.S.) **5**, no. 4(38), 179–182 (1950). (Russian)

This is a very condensed summary of the author's thesis, dealing with operational calculus, operational solution of differential equations, numerical computation of multiple integrals, and differential operators in functional spaces. Many of the results seem to have been described in more detail in a series of papers [Doklady Akad. Nauk SSSR (N.S.) **56**, 779–782 (1947); **62**, 445–447 (1948); Uspehi Matem. Nauk (N.S.) **2**, no. 6(22), 72–158 (1947); **3**, no. 2(24), 234–237 (1948); Trav. Inst. Math. Stekloff **20**, 77–86 (1947); Akušskiĭ and Ditkin, ibid., 7–38 (1947); these Rev. **9**, 353; **10**, 331, 294, 49; **9**, 472, 473]. *R. P. Boas, Jr.*

Polynomials, Polynomial Approximations

Sharma, A. On the zeros of a class of functions. Proc. Benares Math. Soc. (N.S.) **8**, no. 2, 1–21 (1946).

This paper deals with the class C of real functions $f(x)$ of the real variable x which for all negative x exist and are positive together with their derivatives of all orders and which for $|x|<r$ may be represented by the series $f(x)=\sum_0^\infty c_k x^k$. The corresponding polynomial

$$K_{p,n}(x,x_1)=\sum_0^{n-1}c_{p+k}x^k+[(n+p)!]^{-1}f^{(n+p)}(x_1)x^n,$$

is shown to have, when $x_1<0$, at most one real zero in the interval $nx_1\leqq x\leqq r$, and each of its derivatives to have at most one real zero in the interval $x_1\leqq x\leqq r$. Likewise, the polynomial $K_{p,n}(x)=K_{p,n}(x,(n+1)^{-1}x)$ is shown to have at most one zero on the interval $-\infty<x<r$. The method of proof consists in first establishing each theorem in the case $p=0$ and then showing $K_{p,n}(x,x_1)$ and $K_{p,n}(x)$ to be the $K_{0,n}(x,x_1)$ and $K_{0,n}(x)$ for the polynomial

$$f_{p-1}=[f(x)-c_0-c_1x-\cdots-c_{p-1}x^{p-1}]x^{-p},$$

a polynomial also of class C. All functions $f(x)$ of class C are shown to have the further property that for

$$x_0<x_1<\cdots<x_n\leqq 0$$

the Lagrange interpolation polynomial

$$P_n(x)=\sum_0^n[f(x_k)w(x)]/[(x-x_k)w'(x_k)]=\sum_0^n a_kx^k$$

with $w(x)=(x-x_0)(x-x_1)\cdots(x-x_n)$ has all its coefficients a_k positive, and in the interval $-\infty<x<x_0$ has no zero if n is even and at most one zero if n is odd. *M. Marden.*

Bernšteĭn, S. N. On new investigations relative to the best approximation of continuous functions by polynomials. Uspehi Matem. Nauk (N.S.) **5**, no. 4(38), 121–131 (1950). (Russian)

Translation of Proc. fifth Internat. Congr. Math., Cambridge, 1912, vol. 1, pp. 256–266 (1912).

Videnskiĭ, V. S. On an estimate for the derivatives of a polynomial. Doklady Akad. Nauk SSSR (N.S.) **73**, 257–259 (1950). (Russian)

Let $K_n(x)$ and $L_n(x)$ be real polynomials of degree n, positive at 1, with all zeros of $(1+x)^{\frac{1}{2}}K_n(x)$ and $(1-x)^{\frac{1}{2}}L_n(x)$ lying in $(-1, 1)$ and separating each other. By reasoning similar to that of his earlier note [same Doklady (N.S.) **67**, 777–780 (1949); these Rev. **11**, 104] the author proves that if $P_n(x)$ is a polynomial of degree not exceeding n such that on $[-1, 1]$

$$(*)\qquad |P_n(x)|^2\leqq(1+x)K_n^2(x)+(1-x)L_n^2(x),$$

then

$$|P_n^{(k)}(x)|\leqq|D^k\{(1+x)^{\frac{1}{2}}K_n(x)+i(1-x)^{\frac{1}{2}}L_n(x)\}|,$$

where $D=d/dx$. The inequalities of the earlier note were inspired by the representation of a nonnegative polynomial on $[-1, 1]$ as $M_n^2(x)+(1-x^2)N_{n-1}(x)$; those of the present note, by the representation $(1+x)K_n^2(x)+(1-x)L_n^2(x)$. A further set of inequalities arising from the representation $A^2(x)+xB^2(x)$ on $(0, \infty)$ are also given. In addition, if (*) is satisfied and x_j are the zeros of $D^k\{(1+x)^{\frac{1}{2}}K_n(x)\}$, y_j those of $D^k\{(1-x)^{\frac{1}{2}}L_n(x)\}$, arranged in increasing order,

then $|P_n^{(k)}(x)| \leqq |D^k\{(1+x)^{k}K_n(x)\}|$ for $x > y_{n+1-k}$ and $|P_n^{(k)}(x)| \leqq |D^k\{(1-x)^{k}L_n(x)\}|$ for $x < x_1$. Similar results are given in terms of M_n and N_{n-1} of the previous note and in terms of A and B. *R. P. Boas, Jr.*

Tricomi, Francesco G. Sul comportamento asintotico dei polinomi di Laguerre. Ann. Mat. Pura Appl. (4) **28**, 263–289 (1949).

The behavior of generalized Laguerre polynomials

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}$$

has been investigated by many authors. In the present memoir the author gives, for the first time, a set of four asymptotic expansions which together cover the entire positive real axis. He puts $\nu = 4n+\alpha+2$ and distinguishes four regions: (I) the neighborhood of the origin, more precisely $x > 0$, $x = O(n^{-1})$; (II) the oscillatory region $0 < x < \nu$ which contains all zeros; (III) the neighborhood of ν, more precisely $x = \nu + O(n^{\frac{1}{3}})$; (IV) the monotonic region $x > \nu$. In (I), an expansion of $x^{\frac{1}{4}\alpha}e^{-\frac{1}{2}x}L_n^{(\alpha)}(x)$ in a series of Bessel functions [cf. also the author, same Ann. (4) **26**, 141–175 (1947); these Rev. **10**, 605] provides an asymptotic expansion. In (II) and (IV), the saddle point method can be applied to an integral representation of $L_n^{(\alpha)}$. The behavior in (III) has already been investigated by the author [Comment. Math. Helv. **22**, 150–167 (1949); these Rev. **10**, 703]. In each of the four cases the author tests his expansions numerically, comparing exact values of $e^{-\frac{1}{2}x}L_n^{(\alpha)}(x)$ with values computed from his series. The regions of validity overlap, and the author gives a (numerical) comparison of accuracy of his expansions in the overlap intervals. By using a general theorem developed in an earlier paper [same Ann. (4) **26**, 283–300 (1947); these Rev. **10**, 700] the author is able to give asymptotic formulas for the zeros of Laguerre polynomials. These formulas depend on the roots of the equation $x + \sin x = c$ which are tabulated in the paper.

A. Erdélyi (Pasadena, Calif.).

Toscano, Letterio. Relazione integrale di trasformazione dei polinomi ultrasferici in quelli di Laguerre. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 200–202 (1950).

The generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ are expressed in terms of the ultraspherical polynomials $V_n^{(\lambda)}(x)$. [The notation of $L_n^{(\alpha)}(x)$ is the same as in Szegö, Orthogonal polynomials [Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939; these Rev. 1, 14]; as to $V_n^{(\lambda)}(x)$, we have $V_n^{(\lambda)}(x) = P_n^{(\lambda)}(x)$, $s = 2\lambda$.] The starting point is a formula of Uspensky representing $L_n^{(\alpha)}$ in the form of an integral involving Hermite polynomials H_n [cf. Szegö, loc. cit., p. 103, (5, 6.5)]. Expressing H_n as an integral and after suitable transformations of hypergeometric functions, the following formula arises:

$$L_n^{(\alpha)}(x) = \frac{2\pi^{-\frac{1}{2}}}{(\alpha+\frac{1}{2})(\alpha+\frac{3}{2}) \cdots (\alpha+n-\frac{1}{2})} x^{n+\frac{1}{2}} \times \int_0^{\pi/2} e^{-x\cos^2\theta} (\sin\theta)^{-2n-2} V_{2n}^{(2n+1)}(\cos\theta)\, d\theta.$$

G. Szegö (Stanford University, Calif.).

Bagchi, Haridas, and Chakrabarti, Nalinikanta. Note on Laguerre's polynomial $L_n(z)$ and its associated equations (functional and differential). Bull. Calcutta Math. Soc. **42**, 17–24 (1950).

Bagchi, Haridas, and Chakrabarti, Nalini Kanta. Note on a triad of functional equations connected with the Laguerre's polynomial $L_n(z)$. Bull. Calcutta Math. Soc. **42**, 57–60 (1950).

The authors' work is prompted by the observation that Laguerre polynomials satisfy the three functional equations

(i) $$f'_n(z) = n(f'_{n-1}(z) - f_{n-1}(z)),$$

(ii) $$f_{n+1}(z) - (2n+1-z)f_n(z) + n^2 f_{n-1}(z) = 0,$$

(iii) $$zf'_n(z) = nf_n(z) - n^2 f_{n-1}(z),$$

and the differential equation

(A) $$zf_n''(z) + (1-z)f_n'(z) + nf_n(z) = 0.$$

In the first paper it is shown that (i) and (ii) imply (A), and the generating function of the f_n is obtained. In the second paper it is pointed out that any two of the four relations imply the other two. *A. Erdélyi.*

Srivastava, V. N. L. On generalised Legendre polynomials. Bull. Calcutta Math. Soc. **42**, 25–30 (1950).

The polynomials have been introduced by Sharma [same Bull. **40**, 195–206 (1948); these Rev. **10**, 703]. The present paper contains the evaluation of many integrals involving these polynomials. *A. Erdélyi* (Pasadena, Calif.).

Special Functions

Poli, L. Tangentes d'ordre supérieur et nombres de Bernoulli généralisés. Ann. Univ. Lyon. Sect. A. (3) **12**, 5–25 (1949).

The hyperbolic sines of the third order are three independent solutions of the differential equation $y''' = 1$, namely,

$$h_1(x) = \sum x^{3n+2}/(3n+2)!, \quad h_2(x) = \sum x^{3n+1}/(3n+1)!,$$
$$h_3(x) = \sum x^{3n}/(3n)!.$$

Then there are two hyperbolic tangents of the third order, which may be expressed symbolically by

$$T(x) = e^{Tx} = \sum T_n x^n/n! = h_1(x)/h_3(x),$$
$$S(x) = e^{Sx} = \sum S_n x^n/n! = h_2(x)/h_3(x).$$

The author also introduces generalized Euler numbers E_n defined by $1/h_3(x) = e^{Ex} = \sum E_n x^n/n!$. The three nonzero sets of constants E_{3n}, T_{3n+2}, and S_{3n+1} are evaluated in terms of determinants, and various relations established between them. Generalized Euler polynomials are defined symbolically by $E_n(x) = (x+E)^n$, and a set of constants a_n determined uniquely from the equation

$$E_n(x) + E_n(x+1) = a_n(x) = (a+x)^n.$$

It is shown that the E_n's are related to the usual Euler polynomials $\bar{E}_n(x)$ by the equation $2E_n(x) = \bar{E}_n(a+x)$. In particular, when $x = 0$ we have $2E_n = \bar{E}_n(a)$, so that when $n \not\equiv 0 \pmod 3$ the a_n's are the roots of the Euler polynomials. A very similar relationship is established between the polynomials $E_n(x)$ and the Bernoulli polynomials.

M. C. Gray (Murray Hill, N. J.).

Kneser, Hellmuth. Die Potenzreihe der reziproken Gammafunktion. Math. Z. **52**, 655–668 (1950).

The author investigates the coefficients c_n in the expansion $1/\Gamma(1-s)=\sum_{n=0}^{\infty}c_nS^n/n!$. He uses Hankel's integral representation of the reciprocal gamma function to obtain an integral for c_n, and applies the method of steepest descents to this integral. His principal result is

$$c_n=(2\log n)^{\frac{1}{2}}(\pi n)^{-1}\Im[p^ne^{-np}(1+o(1))],\quad n\to\infty,$$

where p is the (unique) root of the equation $pe^p=-n$ in the strip $0<\Im p<\pi$. Other results are an expansion of p for large n, and a statement about the distribution of the $+$ and $-$ signs in the sequence $\{c_n\}$. *A. Erdélyi.*

Ceschino, F. Les fonctions hypergéométriques d'ordre supérieur à deux variables. Ann. Soc. Sci. Bruxelles. Sér. I. **64**, 13–21 (1950).

The author investigates the dependence on x and y of

$$H_n=\int_g^h(u-a_1)^{b_1-1}\cdots(u-a_n)^{b_n-1}(u-x)^{\lambda-1}(u-y)^{\mu-1}du,$$

where g, h are two of the quantities $a_1,\cdots,a_n,x,y,\infty$. He derives the system of partial differential equations satisfied by H_n, and various other properties. He comments on the fact that the coefficients in the power series expansion of H_n are expressible in terms of Lauricella's F_D, but does not seem to notice that H_n itself can be so expressed.

A. Erdélyi (Pasadena, Calif.).

Buchholz, Herbert. Die asymptotischen Entwicklungen für die beiden parabolischen Funktionen $M_{\kappa,\mu/2}(z)$ und $W_{\kappa,\mu/2}(z)$ bei grossen Werten von κ und z für

$$-\infty<z/4\kappa<+\infty.$$

Z. Angew. Math. Mech. **30**, 133–148 (1950). (German. English, French, and Russian summaries)

The author applies the method of steepest descents to integral representations obtained from the standard integrals by a change of variable in order to investigate the behavior of the confluent hypergeometric functions mentioned in the title when κ and z are both large and κ/z is real. The investigation is carried out in three cases in which $z/(4\kappa)$ is, respectively, >1, between 0 and 1, and <0. The investigation of the transition region z near 4κ is deferred to a later paper. The first three terms of the asymptotic expansions are given. There are applications to Laguerre polynomials. *A. Erdélyi* (Pasadena, Calif.).

Leitner, A., and Spence, R. D. The oblate spheroidal wave functions. J. Franklin Inst. **249**, 299–321 (1950).

The following quotations are from the introduction: "While a number of authors have discussed aspects of the [oblate spheroidal wave] functions, we feel that a relatively comprehensive analysis of these functions, which now command wide interest, is lacking"; and "The viewpoint adopted here, in the analysis of the functions at small arguments, was previously used by Page [Physical Rev. (2) **65**, 98–110, 111–117 (1944); these Rev. **5**, 276]. In this discussion of the prolate spheroidal functions, simple power series are used. This method is of advantage, in particular for the radial functions, since the series of spherical Bessel functions converge slowly at small arguments [in fact, they diverge if $|\xi|<1$] and do not explicitly show the properties of these functions near the singular points of the differential equations"; and "While not all the material presented is original, it is hoped that it will serve to unify the subject by pointing out the interrelations among the approaches of the various authors in the field". The paper contains expressions for the first five coefficients of a power series for the eigenvalues of spheroidal wave functions. [Simpler expressions have been given by the reviewer [see the following review]. The last line of the reviewer's formula contains a misprint; $(2n+1)^7$ should read $(2n+1)^2$. If this misprint is corrected, the two formulae are essentially identical.] As to the authors' statement "Since the spherical Neumann functions possess a singularity at the origin neither (73), (74), nor (75) and (76) converge well unless $\epsilon\xi$ is considerably greater than unity," the reviewer observes that any of these series diverges inside the unit circle of the complex ξ plane. Typical wave functions representing spherical waves and plane waves are expanded in spheroidal wave functions. Most important are the authors' tables and graphs. The eigenvalues α_{lm} of

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{du_{lm}}{d\eta}\right]-\left[\frac{m^2}{1-\eta^2}-\alpha_{lm}+\epsilon^2(1-\eta^2)\right]u_{lm}=0$$

are given for $m=1$, $l=4(1)8$; $m=2$, $l=2$, $l=4(1)9$; $m=3$, $l=4(1)10$; $m=4$, $l=4(1)11$ as a function of ϵ with $\epsilon=1(1)5$. The number of significant figures varies from 5 to 9. Apart from their graphs the eigenfunctions $u_{lm}(\eta)$ are tabulated for $\eta=0(0.1)1$, $\epsilon=1(1)5$; $m=0$, $l=0(1)5$; $m=1$, $l=1(1)6$; $m=2$, $l=2$, 3; (4D). The norms of most of these functions, as well as the values of certain constants associated with the radial functions, are listed also. *C. J. Bouwkamp.*

Bouwkamp, C. J. On the characteristic values of spheroidal wave functions. Philips Research Rep. **5**, 87–90 (1950).

Spheroidal wave functions are solutions of the differential equation $(1-z^2)y''-2zy'+(\lambda-(m^2/(1-z^2))+k^2z^2)y=0$ which are finite. Given k and m, such solutions exist for certain characteristic values $\lambda_n^m(k)$ of λ. The author gives the expansion of $\lambda_n^m(k)$ in powers of k^2 as far as the term k^8. For $m=1$, $n=1(1)7$ he also gives the coefficients of this expansion as common fractions, the factorization of the denominators of these coefficients, the numerical values as decimal fractions (this table up to k^{10}), and lastly the characteristic values themselves for $k^2=\pm1$, ±4. *A. Erdélyi.*

Bouwkamp, C. J. On the theory of spheroidal wave functions of order zero. Nederl. Akad. Wetensch., Proc. **53**, 931–944 = Indagationes Math. **12**, 326–339 (1950).

Spheroidal wave functions of order zero are solutions of the differential equation (1) $(1-z^2)y''-2zy'+(\lambda+k^2z^2)y=0$ which are finite at $z=\pm1$. The author normalizes these solutions in a manner which is unambiguous when k^2 is real. Let $y(z)$ be such a characteristic solution. It is known that $y(z)$ can be expanded in a series of Legendre polynomials, $y(z)=\sum_{n=0}^{\infty}a_nP_n(z)$. It has sometimes been asserted that then also $\sum a_nQ_n(z)$ is a solution of (1). The author explains why this need not be the case and why the second solution must have the form $Y(z)=\sum a_nQ_n(z)+\sum b_nP_n(z)$. An alternative expansion of the spheroidal function of the first kind is $y(z)=\alpha(kz)^{-\frac{1}{2}}\sum_{n=0}^{\infty}a_nI_{n+\frac{1}{2}}(kz)$. The corresponding second solution of (1) is $Y^*(z)=\alpha^*(kz)^{-\frac{1}{2}}\sum_{n=0}^{\infty}a_nK_{n+\frac{1}{2}}(kz)$, where α and α^* are certain constants. The author also gives the expansion of $e^{ikz}y(z)$ in Legendre polynomials and the corresponding expansion of $e^{ikz}Y^*(z)$ in Legendre functions of the second kind; also the expansion of $e^{ikz}Y^*(z)$ in descending powers of z. He investigates the relation between the three solutions, showing for instance that Y and Y^* differ by a constant multiple of y, and deriving useful integral relation-

ships. The nature of the (logarithmic) singularities of Y and Y^* at $z=\pm 1$ is also elucidated. *A. Erdélyi.*

Bouwkamp, C. J. On integrals occurring in the theory of diffraction of electromagnetic waves by a circular disk. Nederl. Akad. Wetensch., Proc. **53**, 654–661 = Indagationes Math. **12**, 208–215 (1950).

This is a sequel to an earlier paper [same Proc. **52**, 987–994 = Indagationes Math. **11**, 366–373 (1949); these Rev. **11**, 244]. The double integral

$$\int_0^1 P_m^{2n}((1-\rho'^2)^{\frac{1}{2}})(1-\rho'^2)^{-\frac{1}{2}}\rho' d\rho' \times \int_0^{2\pi} \cos(2m\varphi')[\rho^2-2\rho\rho'\cos(\varphi-\varphi')+\rho'^2]^{\frac{1}{2}\mu-\frac{1}{2}}d\varphi',$$

in which n, m, μ are integers, $0\leqq m\leqq n$, $0\leqq\mu$, while $0\leqq\rho\leqq 1$, $0\leqq\varphi\leqq 2\pi$, is evaluated in terms of a hypergeometric series or, alternatively, as a series of associated Legendre functions. The integral was encountered in diffraction theory.

A. Erdélyi (Pasadena, Calif.).

Humbert, Pierre. Fonctions de Bessel et calcul symbolique. Ann. Soc. Sci. Bruxelles. Sér. I. **64**, 55–61 (1950).

In this paper the author shows that in spite of the considerable body of information on the subject indicated in the title, there are aspects of it which have not yet been explored. He first proves a formula announced by him briefly in a recent note [C. R. Acad. Sci. Paris **230**, 504–505 (1950); these Rev. **11**, 433]. Then he shows that the operational image of $x^{\frac{1}{2}\nu}I_\nu(2x^{\frac{1}{2}})$ regarded as a function of x and y is $q(q+\log p)^{-1}e^{p^{-1}}$, and obtains the operational images of $I_\nu(x)$ and of some other functions related to Bessel functions.

A. Erdélyi (Pasadena, Calif.).

Emersleben, Otto. Die elektrostatische Gitterenergie eines neutralen ebenen, insbesondere alternierenden quadratischen Gitters. Z. Physik **127**, 588–609 (1950).

The mathematical appendix gives the definition and the basic properties of Epstein zeta functions. In the body of the paper several special values of zeta functions are obtained; they all refer to zeta functions associated with the quadratic form x^2+y^2. These values are useful in the study of crystal lattices. *A. Erdélyi* (Pasadena, Calif.).

Harmonic Functions, Potential Theory

Frostman, Otto. Potentiel de masses à somme algébrique nulle. Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **20**, no. 4, 1–21 (1950).

The following notations of M. Riesz are used [Acta Litt. Sci. Szeged **9**, 1–42 (1938); 116–118 (1939)]: α-kernel $(H_\alpha(\alpha))^{-1}r^{\alpha-n}$ in the Euclidean n-dimensional space R^n; α-potential U_α^T; α-energy $E_\alpha(T)\geqq 0$; space W_α of the distributions T with finite α-energy. If the distributions T are considered in the sense developed by the reviewer [Théorie des distributions, I. Actualités Sci. Ind., no. 1091, Hermann, Paris, 1950; these Rev. **12**, 31], W_α is complete; the author shows this only at the end of the article. With some modifications and additions by the reviewer, the principal results are the following. In the case $n=3$, $\alpha=4$ [some precautions are necessary since $\alpha>n$; the distributions T must have a global mass $T(1)=0$]. (1) For T to be in W_4, it is necessary and sufficient that $T=-\Delta F$, with $F\in L^2$; then $F=U_2^T$, $\|F\|_{L^2}^2=E_4(T)$. (2) Every measure μ with compact support is in W_4, provided its global mass is 0. These measures are dense in W_4 (the U_2^μ are dense in L^2). If the supports of the μ are contained in a fixed compact set, weak convergence of bounded measures μ has as a consequence their strong convergence in W_4. (3) The author gives two new definitions for the Green function $G(a,x)$ relative to the pole a and the domain D with frontier F where $D+F$ is compact. Then $G(a,x)=U_2^{\delta(a)-\mu_0}$ where the measure μ_0 is defined as follows: (a) if the measure $\mu\geqq 0$ is borne by $D+F$, and satisfies $\mu(1)=1$, $U_2^{\delta(a)-\mu}\geqq 0$ on D, and $=0$ on $R^n-(D+F)$, then the maximum of $E_4(\delta_{(a)}-\mu)$ is attained for $\mu=\mu_0$; (b) if the measure $\mu\geqq 0$ is borne by R^n-D, and satisfies $\mu(1)=1$, $U_2^{\delta(a)-\mu}\geqq 0$, then the minimum of $E_4(\delta_{(a)}-\mu)$ is attained for $\mu=\mu_0$. (4) If the T_i, $i=1,2,3$, are in W_2, the dipolar distribution $T=-\operatorname{div}\mathbf{T}=-\sum_i \partial T_i/\partial x_i$ is in W_4, and $E_4(T)\leqq\sum_i E_2(T_i)$; reciprocally, every distribution in W_4 can be represented in that form. [The author gives a less precise majorization, $E_4(T)\leqq(\pi^2/4)\sum_i E_2(T_i)$, and only when T is a measure; he doesn't give the converse statement.] (5) For $T\in W_4$, the potential U_4^T can be defined up to an additive constant, t is a continuous function, and $-\Delta U_4^T=U_2^T\in L^2$. If T has a compact support, U_4^T can be completely defined, and $U_4^T=-(8\pi)^{-1}r*T$. The author of the article seems to say (page 16) that, for $T\in W_4$, $U_2^T=(4\pi)^{-1}r^{-1}*T$ may be written as $(4\pi)^{-1}T_x(|x-y|^{-1})$; this is false, since $U_2^T\in L^2$ cannot be calculated for an individual value of x. However, $U_4^T=-(8\pi)^{-1}T_x(|x-y|)$. (6) If for a distribution T some constant m exists such that $(T-m\delta)\in W_4$, then $\|T\|^2=E_4(T-m\delta)+|m|^2$. For every compact set there exists a constant C_k such that, if T is borne by K, $\|T\|^2\leqq C_k E_2(T)$. All these results are analogous to those of Deny [see the following review], who considers only the case $\alpha\leqq n$.

L. Schwartz (Nancy).

Deny, Jacques. Les potentiels d'énergie finie. Acta Math. **82**, 107–183 (1950).

If $h(r)=r^{2-p}$ is the "Newtonian kernel" of Euclidean p-space R^p, $p\geqq 3$, then $U^\mu(M)=\int h(\overline{MP})d\mu(P)$ is the Newtonian potential at M due to the positive (i.e., nonnegative) mass distribution μ. The energy of μ is

$$(1)\quad \|\mu\|^2=\int U^\mu(M)d\mu(M)=\int\int h(\overline{MP})d\mu(M)d\mu(P).$$

H. Cartan [Bull. Soc. Math. France **69**, 71–96 (1941); these Rev. **7**, 447] proved the following. If Ω is the space of positive measures of finite energy distributed on a fixed compact set, and if Ω is normed by $\|\mu\|$, then Ω is a complete space. On the other hand, Cartan [Bull. Soc. Math. France **73**, 74–106 (1945); these Rev. **7**, 447] has shown by example that the space Ω^* of those measures σ which are differences of positive measures of finite energy, normed by $\|\sigma\|$, is not a complete space. It is this phenomenon that is the origin of the present paper, although the author is guided by (i) the nonnegativity of the energy integral, and (ii) the principle of the maximum. One of the author's main tasks is to show that if one does complete the space Ω^*, then one obtains not measures, but "generalized distributions" of the type recently introduced by L. Schwartz [Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) **21** (1945), 57–74 (1946); **23**, 7–24 (1948); these Rev. **8**, 264; **10**, 36]. In the Newtonian case, those distributions are related to the magnetic distributions of classical physics. Many of the results hold for $p=2$, but the author believes that the logarithmic potential is worthy of a separate study.

The author begins his paper with a short summary of useful results in the theory of measures and of distributions due to Schwartz [whose notation we use in this review]. Since the Newtonian potential of a positive measure μ may be considered as the product of composition of μ by r^{2-p} (for $p \geqq 3$), it is a natural generalization to consider $U^T = K * T$ as the potential of a distribution T with respect to the kernel K (where K itself is a distribution). The kernel K should (at least) have the property of having nonnegative energy, that is, be of positive type. In addition, U^T should be finite; hence only those distributions T are considered which, for fixed K, have finite energy. If the symbol $\mathfrak{D}$ denotes the set of infinitely differentiable functions φ, each with a compact support in R^p, then a (Schwartz) distribution is a linear form $T(\varphi)$, defined on $\mathfrak{D}$ and subject to a simple continuity requirement. The set $\mathfrak{D}$ is topologized by means of the energy integral to become a space $\mathfrak{D}'$. Multiplications are defined: αT, for $\alpha \in \mathfrak{D}$, $T \in \mathfrak{D}'$, and $S * T$, for S, $T \in \mathfrak{D}$ (where at least one of the factors S, T has a compact support), such that $\alpha T(\varphi) = T(\alpha\varphi)$ and $(S * T)\varphi = S_P\{T_M[\varphi(M+P)]\}$. The space $\mathcal{S}'$ of spherical distributions is the (properly) topologized set of all continuous linear forms defined on the set $\mathcal{S}$ of all functions θ which are infinitely differentiable and which "decrease rapidly at infinity." The distribution $T \in \mathfrak{D}'$ is said to be of positive type if and only if $T(\varphi * \check{\varphi}) \geqq 0$ for all $\varphi \in \mathfrak{D}$. Here $\check{\varphi}(x_1, \cdots, x_p) = \varphi(-x_1, \cdots, -x_p)$.

In chapter I, the author develops a very general potential theory. Let K denote a spherical distribution such that its Fourier transform $\mathfrak{F}(K)$ is a positive measure of "slow growth at infinity." If T is a distribution with a compact support, then $U_K^T = K * T$ is defined to be the potential generated by the distribution T with respect to the kernel K. It is shown that a necessary and sufficient condition that a distribution T, with compact support, vanish identically is that its energy $\|T\|^2 = \int |\mathfrak{F}(U_K^T)|\, d\mathfrak{F}(K)$ vanish. Next the author studies kernels K that satisfy the following hypotheses: $\mathfrak{F}(K)$ and the inverse $1/\mathfrak{F}(K)$ are both positive measures with "slow growth at infinity." The author proves the following extension of a result due to Cartan. The set of distributions T of finite energy, normed by $\|T\|$, is a complete space. It follows that the space of positive measures is complete. Then the author obtains results which are analogues of those in the classical theories of capacity, Dirichlet problems, etc., as well as representation theorems for distributions and potentials of finite energy.

In chapter II, the kernel K is a positive measure, satisfying the hypotheses of chapter I. It follows that $K = \check{K}$, and that the potential (if it exists) is a positive measure: $U^\mu = U_K^\mu = K * \mu$. The author shows that the potential of a distribution of finite energy is a function defined up to a set of exterior capacity zero; in particular, the potential due to a positive measure of finite energy is superharmonic (thus generalizing a well-known result due to F. Riesz). The author then discusses the principle of the maximum and sequences of potentials in the Brelot-Cartan tradition. One result is the following characterization of kernels that yield potentials satisfying the maximum principle. A necessary and sufficient condition that the positive kernel K "satisfy the maximum principle" is that for each pair of positive measures λ, μ of finite energy the equality $\inf(K * \mu, K * \lambda) = K * \nu$ subsist, where ν is a positive measure of finite energy. The inner product used here is that due to Cartan.

Chapter III is devoted to a study of the potentials of order α, i.e., potentials arising from kernels $h(r) = r^{\alpha-p}$, $0 < \alpha < p$; these were introduced and studied by M. Riesz and Frostman [Frostman, Potentiels d'équilibre . . . , Ohlsson, Lund, 1935]. The author shows that in the Newtonian case, $h(r) = r^{2-p}$, $p \geqq 3$, the potentials with finite energy are essentially the functions of class (BL) explicitly studied by Nikodým [Fund. Math. 21, 129–150 (1933)]. This leads at once to an extension and a converse of the theorem due to Evans that every distribution T of finite energy is a magnetic distribution whose components have summable squares. The results of this chapter are extended to polyharmonic potentials.

In the last section, chapter IV, the author studies Newtonian potentials exclusively. The functions of class (BL) in an open region ω possess pseudo-limits, as foretold by Brelot. These pseudo-limits are used to give extensions of recent results due to Beurling [Acta Math. 72, 1–13 (1939); these Rev. 1, 226], and Dufresnoy [Bull. Sci. Math. (2) 69, 21–36 (1945); these Rev. 7, 56] in the case ω is the unit disc in the plane. Finally, the author defines certain potentials F of finite energy, for an open region ω, and shows that these F are of class (BL) in ω; moreover, F constitutes a linear variety that is orthogonal (under a suitable inner product) to the variety H of uniform harmonic functions of class (BL) in ω. This last result has much in common with a recent result due to Ahlfors [Math. Ann. 120, 36–42 (1947); these Rev. 9, 238]. *M. Reade.*

Mineo, Massimo. Sviluppo rigoroso in serie del potenziale newtoniano terrestre. Boll. Un. Mat. Ital. (3) 4, 391–394 (1949).

Pizzetti [Principi della teoria meccanica della figura dei pianeti, Spoerri, Pisa, 1913, p. 52] has given an expression for terrestrial Newtonian potential under the hypothesis that a surface of equilibrium exterior to the earth is an ellipsoid of revolution. In the present note the author achieves a development in the form of a convergent multiple series for this rather involved expression.

E. F. Beckenbach (Los Angeles, Calif.).

Differential Equations

Zwirner, Giuseppe. Un teorema di unicità per gli integrali di un sistema di due equazioni differenziali ordinarie del primo ordine. Ann. Mat. Pura Appl. (4) 29, 327–334 (1949).

This paper contains an existence theorem giving a sufficient condition that the differential system $y' = f(x, y, z)$, $z' = g(x, y, z)$, $y(a) = \alpha$, $z(b) = \beta$ have a unique solution on $a \leqq x \leqq b$; and a comparison theorem giving a sufficient condition that $y_1'' = f(x, y_1, y_1')$, $y_2'' = g(x, y_2, y_2')$, $y_1(a) \leqq y_2(a)$, $y_1(b) \leqq y_2(b)$ imply $y_1(x) \leqq y_2(x)$ on $a \leqq x \leqq b$. The existence theorem is designed to include two results of Scorza Dragoni [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 22, 44–48 (1935); Rend. Sem. Mat. Univ. Padova 12, 30–50 (1941); these Rev. 8, 207] and one of Trevisan [Rend. Sem. Mat. Univ. Padova 17, 219–221 (1948); these Rev. 10, 375], and the comparison theorem to include that of Groppi [Boll. Un. Mat. Ital. 17, 179–182 (1938)]. *J. M. Thomas.*

Erugin, N. P. Note on the integration in finite form of a system of two equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 315 (1950). (Russian)

The author points out that the system of two real equations $dx/dt = u(x, y)$, $dy/dt = v(x, y)$ can be reduced to the

complex equation $dz/dt=F(z)$, $z=x+iy$, if u and v satisfy the Cauchy-Riemann conditions, and illustrates this by means of the equation $dz/dt=az+bz^2$. This method may be quite useful in obtaining families of typical solutions.

R. Bellman (Stanford University, Calif.).

Germay, R. H. J. Remarque sur une méthode d'approximations successives pour l'intégration des systèmes linéaires d'équations différentielles; extension à des systèmes normaux de forme générale. II. Ann. Soc. Sci. Bruxelles. Sér. I. 62, 109–113 (1948).

If y_{jk} is the kth approximation to a solution y_j of the system $y_j'=f_j(x, y_1, y_2, y_3)$, $j=1, 2, 3$, then

$$|y_j-y_{jk}|<M\beta^k(x-x_0)^{k+1}/(k+1)!,$$

where β is a polynomial in the coefficients of the Lipschitz condition and M bounds $|f_j|$. A similar bound is given for the derivative.

J. M. Thomas (Durham, N. C.).

Germay, R. H. Sur les équations récurrentes, aux différentielles totales, complètement intégrables. Ann. Soc. Sci. Bruxelles. Sér. I. 64, 62–73 (1950).

The method of successive approximations is applied to extend the usual theorem on completely integrable systems to $dz_k=P_k dx+Q_k dy$, where k runs through the positive integers. The given P_k, Q_k are functions of x, y, z_k, z_{k+1} and together with those of their first derivatives appearing in the integrability conditions are continuous and bounded in absolute value by a single constant.

J. M. Thomas.

Revuz, André. Sur l'intégrabilité du système différentiel

$$\frac{dy_i}{dx}=f_i(x, y_1, y_2, \cdots, y_n).$$

Bull. Tech. Univ. Istanbul 2, 27–40 (1949). (French. Turkish summary)

Let R be the set $(x, y_1, \cdots, y_n)$ for $a<x<b$, $c_i<y_i<d_i$ and E a set whose section by $x=$constant is everywhere dense on the section of R by $x=$the same constant. The paper shows that the differential system mentioned in the title has a solution $y_i(x)$ if the following sufficient conditions are satisfied: (i) the f's are continuous in the y's in R; (ii) from each point $P_0=(x_0, y_{10}, \cdots, y_{n0})$ of E starts an arc $y_i=u_i(x, P_0)$, such that on $x_0\leqq x\leqq x_0+\alpha(P_0)$ the u's are continuous and have an integrable right derivative $u_i'(x)$; (iii) $u_i'(x)$ and $f_i[x, u_i(x, P_0)]$ are continuous at x_0 and have equal values at x_0; (iv) $|u_i'(x)|\leqq M(x)$, where M is integrable on (a, b). The author remarks that this result overlaps those of A. Marchaud [Bull. Soc. Math. France 62, 1–38 (1934)] and gives an example to show that his theorem is not implied by Marchaud's.

J. M. Thomas.

De Gennaro, Antonio. Sul comportamento asintotico degli integrali di certe equazioni differenziali lineari ordinarie. Giorn. Mat. Battaglini (4) 3(79), 49–62 (1950).

The results concerning the perturbed nth order differential equation treated in an earlier paper [cf. same Giorn. (4) 2(78), 42–54 (1948); these Rev. 10, 536] are extended to cover the case of coincidences among the characteristic exponents of the homogeneous unperturbed system, i.e., the system with $Q(t)=0$ and $r(t)=0$ in the notation of the review cited above. [The remark made in the earlier review applies here also. This is most readily seen if it is observed that there is no restriction in assuming the matrix B to be in (Jordan) canonical form.]

N. Levinson.

Krein, M. G. A generalization of some investigations of A. M. Lyapunov on linear differential equations with periodic coefficients. Doklady Akad. Nauk SSSR (N.S.) 73, 445–448 (1950). (Russian)

The author discusses the stability of solutions of the vector differential equation $dx/dt=\lambda JH(t)x$, λ a parameter, $H(t)$ a $2m$ by $2m$ real symmetric periodic matrix, and J is the matrix $\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, where in turn I_m is the m by m identity matrix. Conditions on λ are obtained, which imply that all solutions of the equation are bounded.

J. G. Wendel (New Haven, Conn.).

Solncev, Yu. K. Stability according to Lyapunov for solutions of systems of differential equations with discontinuous right sides. Uspehi Matem. Nauk (N.S.) 5, no. 4(38), 140–141 (1950). (Russian)

The author announces results concerning the Lyapunov and asymptotic stability of stationary solutions of the system $dx/dt=X(x, y)$, $dy/dt=Y(x, y)$. It is apparently intended that X and Y may have discontinuities of the first kind on certain smooth curves, but the conditions actually imposed by the author imply continuity of X and Y. The results state that (under conditions not made precise) questions of stability may be reduced to the consideration of a system in which X and Y are piecewise linear functions, which are allowed to have discontinuities on certain line segments.

J. G. Wendel (New Haven, Conn.).

Lur'e, A. I. On auto-oscillations in some regulating systems. Avtomatika i Telemehanika 8, 335–348 (1947). (Russian)

Lur'e, A. I. On the stability of the auto-oscillations of regulating systems. Avtomatika i Telemehanika 9, 361–362 (1948). (Russian)

The author continues his study of systems of the form

$$\dot{x}_k=-\rho_k x_k+f(x),$$
$$\dot{x}=\sum_k \beta_k x_k-rf(x),$$

where f is essentially a unit step function. [Cf. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 9, 353–367 (1945); Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 651–666 (1948); these Rev. 7, 300; 11, 110.] The second paper is a supplement to the first.

J. G. Wendel.

Malkin, I. G. Oscillations of systems with several degrees of freedom, close to systems of Lyapunov. Amer. Math. Soc. Translation no. 21, 30 pp. (1950).

Translated from Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 673–690 (1948); these Rev. 10, 457.

Malkin, I. G. On Poincaré's theory of periodic solutions. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 633–646 (1949). (Russian)

Consider a real system Σ_μ: $\dot{x}_s=X_s(t, x_1, \cdots, x_n, \mu)$, $s=1, 2, \cdots, n$, where the X_s are continuous with period 2π in t, and have continuous first derivatives in the other variables for (x) in a certain region G and $|\mu|$ small. Given the existence of a family of periodic solutions of the system Σ_0 depending upon k parameters, the author discusses quite fully the existence of periodic solutions of Σ_μ generated by those of Σ_0. [The same questions are dealt with for analytical systems in The Methods of Lyapunov and Poincaré in the Theory of Nonlinear Oscillations, OGIZ, Moscow-Leningrad, 1949; these Rev. 12, 28].

S. Lefschetz.

Malkin, I. G. On the theory of oscillations of quasilinear systems with many degrees of freedom. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 353–370 (1950). (Russian)

In the present paper the author continues to follow the program which he seems to have drawn for himself, to extend to more or less general nonanalytical systems of differential equations results already acquired for analytical systems. He now considers a quasi-linear system of order n: $dx_s/dt=\sum a_{sr}x_r+\mu f_s(t, x_1, \cdots, x_n)$, $s=1, \cdots, n$, where the a_{sr} are constants, μ is a small parameter, and the functions f_s are continuous with period 2π in t and have Fourier expansions. Relative to x_s they have continuous first partial derivatives satisfying a Lipschitz condition in a certain domain. The problem is to find periodic solutions of period 2π which for $\mu=0$ tend to similar solutions of the limiting linear system. In a previous paper [see the preceding review] the author discussed what happens when the linear system has a periodic solution depending upon a certain number of parameters. The results of that paper are applied here with particular emphasis on resonance, i.e., when the matrix of the a_{ij} has a certain number of characteristic roots which are zero, or of the form $\pm mi$, m an integer. The solutions are obtained by successive approximations giving rise to series which are shown to converge. [Further relevant references: Malkin, same journal **14**, 13–22 (1950); these Rev. **12**, 28; A. I. Lur'e, ibid. **12**, 353–362 (1948); these Rev. **10**, 193]. *S. Lefschetz* (Princeton, N. J.).

v. Mises, R. Die Grenzschichte in der Theorie der gewöhnlichen Differentialgleichungen. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 29–34 (1950).

The equation (*) $y'=f(x, y)+\nu g(x, y)y''$ is considered for $x_1\leqq x\leqq x_2$ with f and g continuous and uniformly bounded and $g>0$. It is assumed that $u'=f(x, u)$, $u(x_1)=y_1$, and $v'=f(x, v)$, $v(x_2)=y_2$ have unique solutions $u(x)$ and $v(x)$ for $x_1\leqq x\leqq x_2$. The author proves that if (*) has a solution for small ν satisfying $y(x_1)=y_1$ and $y(x_2)=y_2$ then, as $\nu\to 0+$, $\lim y(x)=u(x)$ uniformly for $x_1\leqq x\leqq x_2-\epsilon$, $\epsilon>0$, and similarly for $v(x)$ when $\nu\to 0-$. *N. Levinson.*

Volosov, V. M. On differential equations with a small parameter in the highest derivative. Doklady Akad. Nauk SSSR (N.S.) **73**, 873–876 (1950). (Russian)

The differential equation (*) $\mu y''+Q(x, y)=0$ is considered for small positive μ, and Q is defined and twice differentiable in a vertical strip, $Q(x, 0)=0$, $yQ(x, y)>0$ for $y\neq 0$, and Q is $O(y)$ for large y. Let y_0 and y_0' be fixed initial conditions, and $y_\mu(x)$ the corresponding solution of (*). Let x_1 and x_2 be successive zeros of $y_\mu(x)$; then x_2-x_1 is $O(\mu^{\frac{1}{2}})$ for small μ. There exist smooth functions $F_1(x)>0>F_2(x)$ independent of μ, such that for sufficiently small μ the maxima (minima) of $y=y_\mu(x)$ are as close as desired to the curve $y=F_1(x)$ ($y=F_2(x)$). Reference is made to the work of Tihonov [Mat. Sbornik N.S. **22**(64), 193–204 (1948); these Rev. **9**, 588]. *J. G. Wendel* (New Haven, Conn.).

Stepanov, V. V. On the solutions of a linear equation with periodic coefficients in the presence of a periodic disturbing force. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 311–312 (1950). (Russian)

Some results concerning resonance are obtained for the equation $\ddot{x}+p(t)x=f(t)$, p and f periodic, by means of Fourier series techniques. A typical theorem is: Let the homogeneous equation have bounded solutions, and let $f(t)=ae^{\Omega it}$, $p(t)=p(t+2\pi)$. Writing $x_1=e^{\lambda it}\varphi_1(t)$, $x_2=e^{-\lambda it}\varphi_2(t)$, φ_1 and φ_2 periodic with period 2π, as a fundamental pair of solutions of the homogeneous equation, where α is not an integer, then resonance can occur only if $\lambda+\alpha$ or $\lambda-\alpha$ is an integer. *J. G. Wendel* (New Haven, Conn.).

Gusarova, R. S. On the boundedness of the solution of a linear differential equation with periodic coefficients. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 313–314 (1950). (Russian)

The author proves by elementary means the following generalization of a theorem of Lyapunov. "If $p(x)\geqq n^2$, $-\infty<x<\infty$, $\pi\int_0^\pi p(x)dx\leqq 4(n+1)+n^2(\frac{1}{2}\pi^2+4)$, $n=0, 1, 2, 3, 4$, then all solutions of $y''+p(x)y=0$ are bounded for $-\infty<x<\infty$." *R. Bellman* (Stanford University, Calif.).

***Strelkov, S. P. Vvedenie v teoriyu kolebaniĭ. [Introduction to the Theory of Vibration].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 344 pp.

This text is addressed to the engineer rather than to the mathematician. Mechanical and electrical systems are considered. While most of the treatment is for linear systems with constant coefficients, nonlinear systems, parametric excitation, and distributed parameters are also discussed. *N. Levinson* (Cambridge, Mass.).

Klotter, K. Das Ausschlag-Zeit-Diagramm einer "einfachen Schwebung." Z. Angew. Math. Mech. **30**, 190 (1950).

The author corrects a statement made by K. W. Wagner [Einführung in die Lehre von den Schwingungen und Wellen, 2d ed., Dieterich, Wiesbaden, 1947] to the effect that the function $\sin\omega_1 t+\sin(\omega_2 t+\alpha)$ exhibits a "phase jump" at certain instants of time t. *E. N. Gilbert.*

Bower, John L. A note on the error coefficients of a servo mechanism. J. Appl. Phys. **21**, 723 (1950).

The author derives an expression for the error of a linear servomechanism in terms of the input signal and certain of its derivatives. A few consequences of the formula are discussed. *L. A. MacColl* (New York, N. Y.).

Miller, Kenneth S., and Schwarz, Ralph J. Analysis of a sampling servo mechanism. J. Appl. Phys. **21**, 290–294 (1950).

The authors treat a servo loop that samples the servo errors periodically and "clamps" these samples in the interim. The solution is found by first solving the open loop with the help of the one-sided Green's function, and matching boundary conditions at the sampling instants. Stability relations similar to those given by MacColl [Fundamental Theory of Servomechanisms, Van Nostrand, New York, 1945; these Rev. **7**, 299] are found. Unfortunately, the authors have used the symbol $\epsilon(t)$ for two distinct functions, namely, the error signal before sampling and the output of the clamping mechanism. *G. R. Stibitz* (Burlington, Vt.).

Marx, Helmut. Über die Differentialgleichung

$$\frac{dy}{dx}=\frac{\alpha x(x+\beta)y}{(x-1)^{\alpha+1}y-x^2(x-1)}.$$

Lösungsverlauf, graphische Integration, Reihenentwicklungen. Mitt. Math. Sem. Univ. Giessen **29**, ii+27 pp. (1943).

The differential equation arises in a wave propagation problem relating to gases, and the values of the con-

stants of interest in that problem are $\alpha=1$ or $4/3$, and $\beta=(2-\alpha)^{-1}(c_p/c_v-1)$, where c_p/c_v has the usual significance. In the first part of this thesis the author discusses the qualitative features of the integral curves of his differential equation, having in mind the stated numerical values of the constants. In the second part he describes the geometry of an instrument which, if it could be built, would perform the mechanical integration of the differential equation. In the last part he obtains power series expansions of the solutions of his equation for the neighborhood of the singularities. In this last part he discusses the more general equation $[w-z^{-\lambda}q(z)]dw/dz=z^{-\lambda-1}wp(z)$, in which p and q are power series in z. *A. Erdélyi* (Pasadena, Calif.).

Hacar Benitez, Miguel Angel. Interpretation of the second order differential equation (1) $kyy''+y'^2+1=0$ and a study of some of its solutions. Gaceta Mat. (1) **2**, 46–53 (1950). (Spanish)

Garcia Araez, Rafael. Rational curves and algebraic differential equations. Gaceta Mat. (1) **2**, 6–14 (1950). (Spanish)

Wachs, Sylvain. Sur l'abaissement de l'ordre d'une équation différentielle linéaire. Bull. Sci. Math. (2) **74**, 40–72 (1950).

Given a linear homogeneous differential equation $F(y)=0$ of order $n+1$, the author defines functions u, f and a linear nonhomogeneous equation $G(y)=f$ so that the respective general solutions have the forms $c_1y_1+\cdots+c_ny_n+c_{n+1}u$ and $c_1y_1+\cdots+c_ny_n+u$. The function $1/f$ satisfies the adjoint of $F(y)=0$ and u satisfies a nonlinear equation of order n called by the author the resolvent (in the case $n=1$, the resolvent becomes the Riccati equation). The treatment is elementary and detailed and, as the author states, the results are for the most part known, although the relations between them are here presented from a new point of view. *J. M. Thomas* (Durham, N. C.).

Mitrinovitch, D. S. Procédé de formation des critères d'intégrabilité des équations différentielles linéaires à coefficients ayant des formes données à l'avance. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **2**, 209–246 (1949). (Serbian. Russian and French summaries)

The author studies the equation

$$(1)\qquad L(y)\equiv\varphi_0y^{(n)}+\cdots+\varphi_ny=0$$

(φ_k functions of x), also the system

$$(2)\qquad f_{k1}y'_{k-1}+f_{k2}y_{k-1}=y_k,\quad f_{n1}y'_{n-1}+f_{n2}y_{n-1}=0$$

($y_0=y$; $k=1,\cdots,n-1$), where (i) $f_{k1}\not\equiv0$, (ii) the f_{kl} are continuous, derivable. Let (3) $\Phi_0y^{(n)}+\cdots+\Phi_ny=0$ be the result of elimination in (2) of $y_1,\cdots,y_{n-1}$. An equation (1) transformable into a system (2) is said to be reducible; (2), (3) are always integrable by quadratures. If (1) is reducible, so is (4) $L(y)=\omega(x)$; then in the second equation (2) 0 is replaced by $\omega(x)$. These ideas are applied to the problem of reducibility of certain particular equations of the second order. *W. J. Trjitzinsky* (Urbana, Ill.).

Mitrinovitch, D. S. Sur une classe d'équations de Riccati invariantes relativement à un changement de fonction. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **2**, 167–186 (1949). (Serbian. Russian and French summaries)

Karapandjitch, G. Généralisation d'une condition de Bougaeff. Bull. Soc. Math. Phys. Serbie **1**, no. 3–4, 117–120 (1949). (Serbian. French summary)

The translation of the Serbian title is: A condition of integrability for Riccati's equation. Using a contact transformation the author modifies a condition due to Bugaev [Mat. Sbornik (1) **17**, 399–438 (1894)]. *W. Feller.*

Gonzalez, Mario O. Memoir on the solution of the ordinary differential equations of the first order which are invariant under contact transformations. Univ. Nac. Tucumán. Revista A. **7**; 109–128 (1949). (Spanish)

The present paper is in the spirit of the classical Lie theory of groups, and is concerned with the solution of first order differential equations (1) $f(x, y, p)=0$, $p=dy/dx$, which are invariant under a one parameter group of contact transformations (2) $X=X(x, y, p; a)$, $Y=Y(x, y, p; a)$, $P=P(x, y, p; a)$. A group of contact transformations (2) is determined by its infinitesimal generator

$$W_p\partial/\partial x+(pW_p-W)\partial/\partial y+(-W_x-pW_y)\partial/\partial p,$$

where $W=W(x, y, p)$ is what Lie calls the characteristic function of the group, and equation (1) is invariant with respect to the group (2) if and only if the function f satisfies the partial differential equation of the first order

$$(3)\quad W_p\partial f/\partial x+(pW_p-W)\partial f/\partial y+(-W_x-pW_y)\partial f/\partial p=0.$$

There are three main sections besides an introduction. The first section contains a list of differential equations of type (1), together with the infinitesimal generators of the corresponding groups of type (2) which leave them invariant, a list which was compiled by choosing W in various simple forms and then integrating the resulting equation (3). The second section shows how, for a certain special class of functions f, the characteristic function of the group (2) leaving (1) invariant may be determined. The third section shows, for the same special class of functions f, for which the group (2) is already known in principle, how equation (1) may be integrated. *J. B. Diaz* (College Park, Md.).

Sears, D. B. Note on the uniqueness of the Green's functions associated with certain differential equations. Canadian J. Math. **2**, 314–325 (1950).

It is proved that the Green's function associated with the differential equation $(d^2\varphi/dx^2)+\{\lambda-q(x)\}\varphi=0$, $0<x<\infty$, is unique (i.e., that the equation is of the limit-point type) under certain conditions, of which the essential one is that $q(x)\geqq-Q(x)$, where $Q(x)$ is positive, and $\int^\infty\{Q(x)\}^{-\frac{1}{2}}dx=\infty$. This completes results of the reviewer [Quart. J. Math., Oxford Ser. (1) **12**, 33–50 (1941); Eigenfunction expansions . . . , Oxford University Press, 1946; same J. **1**, 191–198 (1949); these Rev. **3**, 121; **8**, 458; **10**, 537], Hartman and Wintner [Amer. J. Math. **70**, 295–308 (1948); **71**, 206–213 (1949); these Rev. **10**, 120, 455]. A similar result is obtained for the partial differential equation

$$(\partial^2\varphi/\partial x^2)+(\partial^2\varphi/\partial y^2)+\{\lambda-q(x, y)\}\varphi=0$$

under the condition that $q(x, y)\geqq-Q(r)$, where $x^2+y^2=r^2$, $Q(r)$ is positive, and $\int^\infty\{Q(r)\}^{-\frac{1}{2}}dr=\infty$.

E. C. Titchmarsh (Oxford).

Titchmarsh, E. C. On the discreteness of the spectrum of a differential equation. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 16–18 (1950).

A simple proof is given of the theorem of Weyl that $\phi''+(\lambda-q(x))\phi=0$, $0\leqq x<\infty$, has a discrete spectrum if

$q(x)$ is bounded in any finite interval and tends to infinity as $x \to \infty$. *N. Levinson* (Cambridge, Mass.).

Kodaira, Kunihiko. On ordinary differential equations of any even order and the corresponding eigenfunction expansions. Amer. J. Math. **72**, 502–544 (1950).

In a previous paper [same J. **71**, 921–945 (1949); these Rev. **11**, 438] the author has treated the spectral and expansion theory for a singular ordinary real differential operator of second order by blending theories of Weyl, Stone, and Titchmarsh. In the present paper the corresponding task is undertaken for (formally) self-adjoint ordinary real differential operators of arbitrary even order and analogous results are obtained, for the first time in this general case as far as the reviewer is aware. The method consists in combining the use of the results of the abstract Hilbert space theory (e.g., the spectral theorem) with the development of a suitable generalization of Weyl's theory [see, e.g., Math. Ann. **68**, 220–269 (1910)] in the second order case (e.g., the generalization of Weyl's classification according to the limit circle and limit point case). *E. H. Rothe.*

Greco, Donato. Gli sviluppi in serie di autosoluzioni in un problema ai limiti relativo ad un'equazione differenziale lineare ordinaria del secondo ordine. Giorn. Mat. Battaglini (4) **3(79)**, 86–120 (1950).

The author gives sufficient conditions that a function be represented by the series of characteristic functions of the boundary value problem treated earlier [same Giorn. (4) **2(78)**, 216–237 (1949); these Rev. **11**, 437]. Use is made of integral equations with kernels containing a parameter. *N. Levinson* (Cambridge, Mass.).

Germay, R. H. Sur une modalité de la méthode d'intégration par approximations successives des équations aux dérivées partielles du premier ordre. Ann. Soc. Sci. Bruxelles. Sér. I. **63**, 31–35 (1949).

A solution of the equation $p_1 = f(x_1, \cdots, x_n, z, p_2, \cdots, p_n)$ is found as the limit of a sequence resulting by elimination from successive approximations to integrals of the equations of the characteristics. *J. M. Thomas* (Durham, N. C.).

Germay, R. H. Sur l'intégration par approximations successives de certains systèmes d'équations aux dérivées partielles du second ordre à deux variables indépendantes. Ann. Soc. Sci. Bruxelles. Sér. I. **62**, 3–10 (1948).

Integration of the partial differential system

$$z_{jxy} = a_{jk}z_{kx} + b_{jk}z_{ky} + c_{jk}z_k,$$

where $j, k = 1, 2, \cdots, n$, k is summed, and a, b, c are functions of the independent variables x, y, can be accomplished by applying successive approximations to systems of the form $z_{xy} = \lambda(az_x + bz_y + cz) + d$, each system involving only one unknown z and λ being a parameter. *J. M. Thomas* (Durham, N. C.).

Germay, R. H. Sur une méthode d'approximations successives pour l'intégration de certains systèmes non linéaires d'équations aux dérivées partielles du second ordre à deux variables indépendantes. Ann. Soc. Sci. Bruxelles. Sér. I. **62**, 61–66 (1948).

This paper applies the method of successive approximations to prove the existence of a solution for the partial differential system $z_{jxy} = f_j(x, y; z_1, \cdots, z_k; z_{1x}, \cdots, z_{kx}, z_{1y}, \cdots, z_{ky})$, $z_j(x, 0) = z_j(0, y) = 0$, where the f's are subjected to certain restrictions. *J. M. Thomas* (Durham, N. C.).

Fichera, Gaetano. Risultati concernenti la risoluzione delle equazioni funzionali lineari dovuti all'Istituto Nazionale per le applicazioni del calcolo. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis.-Mat. Nat. (8) **3**, 1–81 = Consiglio Naz. Ricerche Pubbl. Ist. Appl. Calcolo no. **271** (1950).

L'auteur résume sans démonstrations l'ensemble des travaux faits, sur les équations fonctionnelles linéaires par l'Institut de Calcul dirigé par M. Picone, fondé à Naples en 1927 et devenu l'Istituto Nazionale per le Applicazioni del Calcolo, à Rome en 1932. La bibliographie, presque uniquement italienne et concernant surtout les membres de cet Institut, comprend 148 articles de Picone, Fichera, Amerio, Folotti, Ghizetti, Miranda, Faedo, Grioli, etc. On ne peut guère ici que donner une indication des sujets traités: chapitre I, théorèmes de majoration d'intégrales pour des équations différentielles ou aux dérivées partielles de type elliptique ou parabolique, entraînant des résultats d'unicité; chapitre II, méthodes de minimum; méthode des moyennes pondérées ou quadratiques, variationnelle ou de Ritz avec diverses applications (équation de Fredholm, problèmes aux limites, vibrations, . . .); chapitres III et IV, méthode des transformations fonctionnelles, transformations de Laplace et Fourier, fonctions polyharmoniques; chapitre V, traduction d'une équation fonctionnelle linéaire en équations de Fischer-Riesz, théorèmes de systèmes complets relatifs aux fonctions harmoniques et biharmoniques; chapitre VI, autres recherches sur les problèmes aux limites pour les équations linéaires aux dérivées partielles. *M. Brelot* (Grenoble).

Zwirner, Giuseppe. Sull'integrazione di un sistema di equazioni differenziali alle derivate parziali. Ann. Triestini. Sez. 2 (4) **1(17)**, 41–52 (1947).

The author generalizes Laplace's method of transformations for a partial differential equation of the form

$$\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0 \tag{1}$$

[see Darboux, Leçons sur la théorie générale des surfaces . . . , vol. II, Gauthier-Villars, Paris, 1889] to systems of equations. It is now assumed that a, b, and c are square matrices of order n with elements depending on x, y, and that z is a square matrix whose columns represent solutions of a system of equations. With (1) are associated the two "quasi-invariant" matrices $h = \partial a/\partial x + ba - c$, $k = \partial b/\partial y + ab - c$. Under an arbitrary substitution $z = z^*$, (1) is transformed into an equation for z^* with quasi-invariants $h^* = \alpha^{-1}h\alpha$, $k^* = \alpha^{-1}k\alpha$. If either h or k vanishes, the solution of (1) can be reduced to the integration of 2 systems of ordinary differential equations of order n (which for $n = 1$ can be achieved by quadratures). For $h = k$ (1) can be transformed into an equation with $a^* = b^* = 0$. The transformation (2) $z_1 = \partial z/\partial y + az$ also preserves the form of (1). Repeated application of the substitution (2) can result in an equation with a vanishing quasi-invariant, and hence can be used to reduce (1) to the solution of ordinary differential equations. *F. John* (Los Angeles, Calif.).

* **Chen, Yu Why. The characteristic initial value problem for a hyperbolic differential equation with a singularity.** National Peking University Semi-Centennial Volume, Mathematical, Physical and Biological Series, pp. 18–25, 1948.

The author determines for positive σ, τ the solution of the equation

$$u_{\sigma\tau} + \tfrac{1}{2}k(\sigma+\tau)^{-1}(u_\sigma + u_\tau) + l(\sigma+\tau)^{-2}u = \phi(\sigma, \tau, u, u_\sigma, u_\tau) \tag{1a}$$

satisfying (1b) $u(\sigma, 0)=f(\sigma)$, $u(0, \tau)=g(\tau)$. It is assumed that the constants k, l are such that $(k-1)^2-4l\geqq 0$. The linear equation obtained from (1a) by putting $\phi=0$ has a Riemann function $R(\sigma, \tau; \xi, \eta)$ that can be expressed in terms of hypergeometric functions. With the help of R, the boundary value problem for the general equation (1a) can be reduced to the determination of a solution $U(\xi, \eta)$ of an integral equation of the form

$$(2)\quad U(\xi, \eta)=\int_0^t\int_0^x \psi(\sigma, \tau, U, U_\sigma, U_\tau)R(\sigma, \tau; \xi, \eta)d\sigma\, d\tau.$$

Equation (2) can be solved by iteration under suitable regularity assumptions on f, g, ϕ. The solution of $(1, a, b)$ by this method is made difficult by the singular behaviour of R for (σ, τ) near the origin. *F. John.*

Hartman, Philip, and Wintner, Aurel. On the solutions of the equation of heat conduction. Amer. J. Math. **72**, 367–395 (1950).

The object of this paper is to establish some properties for solutions $u(x, t)$ of the heat equation (1): $u_{xx}=u_t$ on the open square $S: 0<x<1, 0<t<1$ which are analogues of well-known properties of functions harmonic in the unit circle. The main results are as follows. (A) Let $f_i(s)$, $i=1, 2, 3$, be bounded measurable functions on $0<s<1$. Let $\theta(x, t)$ be the Jacobi ϑ_3-function (with suitable changes in the units of the independent variables). Then the functions $u_1(x, t)$, $u_2(x, t)$, $u_3(x, t)$ defined by the integrals

$$(2)\quad -\int_0^t \theta_x(x, t-s)f_1(s)ds,\quad \int_0^t \theta_x(1-x, 1-t)f_2(s)ds,$$
$$\tfrac{1}{2}\int_0^1 \{\theta(x-s, t)-\theta(x+s, t)\}f_3(s)ds,$$

respectively, exist as Lebesgue integrals and their sum $u(x, t)$ is a bounded continuous solution of (1) on S for which almost everywhere

$$(3)\quad \lim_{x\to 0^+} u(x, s)=f_1(s),\quad \lim_{x\to 1^-} u(x, s)=f_2(s),$$
$$\lim_{t\to 0^+} u(s, t)=f_3(s).$$

Conversely, if $u(x, t)$ is a bounded continuous solution of (1), then the limits (3) exist almost everywhere, and $u(x, t)=\sum_i u_i(x, t)$ where the functions f_i appearing in the definition of the u_i are given by (3). A discussion of (A) leads to a more satisfactory statement concerning the uniqueness question connected with (1) than has been given hitherto.

To state the remaining results we replace the integrals (2) by Stieltjes integrals, i.e., we replace $f_i(s)ds$ by $dF_i(s)$ in (2) where the $F_i(s)$ satisfy the following assumptions H: F_1 and F_2 are of bounded variation, while for F_3 only $\int_0^1 s(1-s)|dF_3(s)|<\infty$ is assumed. (B) If F_1, F_2, F_3 satisfy H and are nondecreasing in $0\leqq s<1$, $0\leqq s<1$, $0<s<1$, respectively, then $u(x, t)=\sum_i u_i(x, t)$ is a nonnegative continuous solution of (1) on S. Conversely, if $u(x, t)$ is a continuous nonnegative solution of (1) on S, there exist nondecreasing functions $F_i(s)$ satisfying H, and $u(x, t)=\sum_i u_i(x, t)$. (C) Let $F_i(s)$ satisfy H. Then $u(x, t)=\sum_i u_i(x, t)$ is a continuous solution of (1). Moreover the lim sup as $x\to 0^+$ and as $x\to 1^-$ of $\int_0^t |u(x, s)|ds$ are finite; likewise

$$\limsup_{t\to 0^+}\int_0^1 s(1-s)|u(s, t)|ds<\infty.$$

Conversely, if $u(x, t)$ is a continuous solution of (1) satisfying the finiteness conditions just mentioned, then there exist functions $F_i(s)$ satisfying H, and $u(x, t)=\sum_i u_i(x, t)$. On their continuity points the F_i are uniquely determined (up to additive constants). The jumps $F_1(0^+)-F_1(0)$ and $F_2(0^+)-F_2(0)$ are also uniquely determined.

E. H. Rothe (Ann Arbor, Mich.).

Cooper, J. L. B. The uniqueness of the solution of the equation of heat conduction. J. London Math. Soc. **25**, 173–180 (1950).

Let $u(x, t)$ be a solution of the heat equation $u_{xx}=u_t$ for $0<t\leqq T$, which together with u_t and u_{xx} is summable over every finite rectangle $a\leqq x\leqq b$, $T\geqq t\geqq\delta>0$. Also assume for some constant c there exists a sequence $\{x_i\}$ tending to infinity such that $|u(x_i, t)|<e^{cx_i^2}$, $0<t\leqq T$ (a similar condition holding for some sequence $\{x_i'\}$ tending to minus infinity). Under these conditions the author is able to show $u(x, t)=0$, $0<t\leqq T$, if it satisfies the local condition $\lim_{t\to 0}\int_a^b g(x)u(x, t)dx=0$ for every finite interval (a, b) and every summable function $g(x)$ bounded in (a, b).

F. G. Dressel (Durham, N. C.).

Datzeff, Assène. Sur certaines analogies mécaniques de la théorie de la chaleur. C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 2–3, 25–28 (1949).

From the solution of the equation of diffusion $u_t=a^2u_{xx}$ in a prism of infinite length under the condition $u(x, 0)=f(x)$, $-\infty<x<\infty$, the author shows that the center of gravity of the diffusing material maintains a fixed position independent of the time t. The moment of inertia of the material with respect to the plane $x=0$ is equal to that with respect to the centroidal plane plus L^2M where L is the distance between the planes and M is the total mass of the diffusing material. Corresponding properties are given for three-dimensional diffusion. *R. V. Churchill.*

Dacev, A. B. On a general linear problem of heat conduction in a stratified medium. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **14**, 113–127 (1950). (Russian)

Conduction of heat in a medium composed of plane strata is studied in this article. The initial temperature is prescribed, and equal temperature and conservation of heat at interfaces is required. The boundary conditions

$$(A)\quad \begin{aligned}&[\partial U_1/\partial x]_{x_0}+\alpha_1(t)U_1(x_0, t)+\beta_1(t)=0,\\ &[\partial U_n/\partial x]_{x_n}+\alpha_2(t)U_n(x_n, t)+\beta_2(t)=0\end{aligned}$$

are novel in that the coefficients $\alpha_1(t)$ and $\alpha_2(t)$ are variable. In solving the problem, the author replaces boundary conditions (A) by the conditions $U_1(x_0, t)=\phi_0(t)$, $U_n(x_n, t)=\phi_n(t)$, yielding a problem whose solution was given earlier by the author [Doklady Akad. Nauk SSSR (N.S.) **56**, 355–358 (1947); these Rev. **9**, 146]. Then $\phi_0(t)$ and $\phi_n(t)$ are found to satisfy a pair of integral equations of Volterra's second type. The solution so found is verified, and its uniqueness proved. *R. E. Gaskell* (Ames, Iowa).

Datzeff, Assène. Sur le problème linéaire général de propagation de la chaleur dans un milieu à plusieurs couches. C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 2–3, 21–24 (1949).

Let $u_i(x, t)$ denote the temperatures in the ith layer of a conducting medium consisting of n layers $x_{i-1}\leqq x\leqq x_i$, $i=1, 2, \cdots, n$, of different materials initially at temperatures $\Phi_i(x)$. At the faces $x=0$ and $x=x_n$ the temperature functions satisfy conditions of the type $u_x+\alpha u+\beta=0$ where α and β are prescribed functions of the time t. The author

outlines a method of finding the functions u_i as the solutions of a system of Volterra integral equations involving elliptic theta functions. *R. V. Churchill* (Ann Arbor, Mich.).

Jaeger, J. C. Conduction of heat in composite slabs. Quart. Appl. Math. **8**, 187–198 (1950).

Transient one-dimensional heat flow through composite slabs of n layers is compared to the steady flow. The total flow of heat $Q(t)$ through unit area of an outer face of the slab up to the time t is used as the means of comparison. For simple initial and boundary conditions the author sets up a sensible measure of time-lag τ, such that after the time $t=\tau$, $Q(t)$ is approximately the same as the quantity of flow under steady-state conditions. With the aid of Laplace transforms a systematic scheme is devised for computing time-lags for either of the outer faces. Numerical examples are given. The problem arose in connection with the design of the walls of refrigerated railway cars. Other applications are apparent. The complexity of actual formulas for the transient temperatures in slabs of several layers makes the direct use of such formulas impractical and calls for an analysis of the type introduced here.

R. V. Churchill (Ann Arbor, Mich.).

Zanobetti, Dino. Sulla distribuzione transitoria disuniforme di una corrente continua in una lamiera e in un cilindro, e sulla resistenza di rotaia. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 129–134 (1950).

Essentially, the author solves the following two boundary value problems by Laplace transform methods:

(i) $G_{xx}(x,t)-kG_t(x,t)=0,\ -x_0<x<x_0,\ t>0;$
$G(\pm x_0,t)=G_1,\ t>0;\quad G(x,0)=G_0,\ -x_0<x<x_0;$

(ii) $G_{rr}(r,t)+r^{-1}G_r(r,t)-kG_t(r,t)=0,\ 0\leqq r<a,\ t>0;$
$G(a,t)=G_1,\ t>0;\quad G(r,0)=G_0,\ 0\leqq r<a.$

(Subscripts indicate partial differentiation.) In each case the inversion integral is evaluated in terms of residues, and graphs give numerical results in terms of dimensionless variables. *A. Erdélyi* (Pasadena, Calif.).

Krzyżański, Miroslaw. Sur la solution élémentaire de l'équation de la chaleur. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 193–199 (1950).

Let Γ be the region $-\infty<x<\infty$, $0\leqq y\leqq h$. Let $u(x,y)$ be a function continuous in Γ, the point $(0,0)$ excepted, further let u be of class C^2 interior to Γ and there satisfy the heat equation $u_{xx}=u_y$. By imposing on $u(x,y)$ the three additional conditions (1) $u(x,y)\geqq 0$, (2) $u(x,0)=0$, $x\neq 0$, (3) the function $j(y)=(1/2\sqrt{\pi})\int_{-\infty}^{\infty}u(x,y)dx$ is bounded and $j(y)\to 1$ as $y\to 0$, the author is able to show that $u(x,y)$ is necessarily the elementary solution $y^{-\frac{1}{2}}\exp(-x^2/4y)$ of the heat equation. *F. G. Dressel* (Durham, N. C.).

Krzyżański, M. Sur les solutions de l'équation linéaire du type elliptique, discontinues sur la frontière du domaine de leur existence. Studia Math. **11**, 95–125 (1949).

Let $\mathrm{Fr}(D)$ designate the boundary of the bounded domain D, $\bar{D}$ its closure $D+\mathrm{Fr}(D)$, and E a closed set $\subset\mathrm{Fr}(D)$. Let $\{D_i\}$ be a sequence of open domains such that $D_i\subset D_{i+1}\subset D$, $\bar{D}_i\subset\bar{D}-E$, and $D=\lim_{i\to\infty}D_i$. Assume from now on that the Dirichlet problem, u_i a solution of the elliptic equation

$$(*)\quad E(u)\equiv\sum_{i,j=1}^{n}a_{ij}(P)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{j=1}^{n}b_j(P)\frac{\partial u}{\partial x_j}+c(P)u=f(P),$$

$P=P(x_1,\cdots,x_n)$, interior to D_i and $u_i=\Phi_i(P)$ on the boundary of D_i, is solvable for every D_i. The present paper is concerned with the existence of solutions of (*) in D which on $\mathrm{Fr}(D)-E$ assume assigned continuous values $\Phi(P)$. The author terms this the generalized Dirichlet problem for D. The following is a representation of the type of results obtained. With the coefficients a_{ij}, b_j, c, f in (*) continuous, if there exists a function $H(P,k)>0$ of class C^2 for $P\subset\bar{D}-E$ for each value of the parameter k, $0<k<k_0$, having the properties (a) $\lim_{P\to B}H(P,k)=\infty$, (b) $H(P,k_1)/H(P,k_2)\to 0$ as $P\to E$, $k_1<k_2$, (c) $E(H(P,k))<-\delta(k)H(P,k)<0$, (d) there exists a constant M such that $|\Phi(P)|\leqq M\cdot H(P,k)$, $P\subset\mathrm{Fr}(D)-E$, and $|f(P)|\leqq M\cdot H(P,k)$, $P\subset D$, then there exists a solution of the generalized Dirichlet problem for the region D. This solution is unique in the class of functions $u(P)$ such that $|u(P)|<M'\cdot H(P,k')$, where M', k' are positive constants.

In case E consists of a single point Q and the a_{ij} satisfy at Q a Lipschitz condition, the following generalization of a theorem of Zaremba [Bull. Internat. Acad. Sci. Cracovie. Cl. Sci. Math. Nat. **1909**$_1$, 561–563] is given. When $c\leqq 0$, the only solution of the generalized Dirichlet problem $E(u)=0$ in D, $u=0$ on $\mathrm{Fr}(D)-Q$, is $u\equiv 0$ if $\lim_{r\to 0}u(P)r^{n-2}=0$ $(n>2)$, or $\lim_{r\to 0}u(P)(\log(2d/r))^{-1}=0$ $(n=2)$. Here d is the diameter of the set D and r is the distance from P to Q. The author points out that the restriction $c\leqq 0$ can be slightly relaxed in certain cases. *F. G. Dressel.*

Prakash, Surya. Solution of generalized wave equations of order higher than 2. I. Proc. Indian Acad. Sci., Sect. A. **31**, 240–260 (1950).

The generalized wave equation considered in this paper is of the form

$$\prod_{r=1}^{n}\left\{\frac{\partial^2}{\partial x_0^2}-\sum_{s=1}^{m}\frac{\partial^2}{\partial x_s^2}+\chi_r^2\right\}^{p_r}\psi=f(x_0,x_1,\cdots,x_m),$$

where the χ_r are constants, the p_r positive integers, and f a given function. A particular solution is obtained by the method of Fourier transforms, and two cases, viz. (i) all the χ_r different and all the p_r unity, (ii) all the χ_r equal, are discussed in greater detail. *E. T. Copson* (Dundee).

Difference Equations, Special Functional Equations

Shah, S. M. On real continuous solutions of algebraic difference equations. II. Proc. Nat. Inst. Sci. India **16**, 11–17 (1950).

If y is a real continuous solution of the algebraic difference equation of the first order

$$(1)\quad P(y(x+1),y(x),x)=\sum a x^{\alpha}y(x)^{\beta_0}y(x+1)^{\beta_1}=0,$$

then $\liminf_{x\to\infty}\log\log(|y(x)|/|x|)<\infty$, which cannot be improved [see the author, Bull. Amer. Math. Soc. **53**, 548–558 (1947); these Rev. **9**, 189]. If $y(x)$ is monotone for $x>x_0$, then $\limsup\log\log|y(x)/x|<\infty$. Some of the results are as follows. If the (real continuous) solution $y(x)$ of (1) satisfies (2) $y(x+h)\geqq y(x)/e_1(Bx)$ $(0\leqq h\leqq 1,\ x>x_0;\ B>0)$, then $y(x)<e_2(Ax)$ for all $x>x_0(A)$ and some $A>B$. Let (3) $P(y(x+m),\cdots,y(x),x)=\sum ax^{\alpha}y(x)^{\beta_0}\cdots y(x+m)^{\beta_m}=0$ be an algebraic difference equation of order m and $T'=a'x^{\alpha'}\cdots y(x+m)^{\beta'_m}$ be the principal term; suppose that, when $\beta'_m=\beta_m$, one has $\beta'_{m-1}>\beta_{m-1}$ or $\beta'_{m-1}=\beta_{m-1}$ and

$\beta'_i \geqq \beta_i$, $i=0, \cdots, m-2$; if $y(x)$ is a solution of (3) satisfying (2), such that $\log y(x)/\log x \to \infty$ (as $x \to \infty$), then $y(x) < e_3(Ax)$ for all $x > x_0(A)$ and some $A > B$.

W. J. Trjitzinsky (Urbana, Ill.).

Myškis, A. D. Linear homogeneous differential equations of the first order with retarded argument. Uspehi Matem. Nauk (N.S.) **5**, no. 2(36), 160–162 (1950). (Russian)

The author classifies the solutions of the equations $y^{(1)}(x) - M(x)y(x-\Delta(x)) = 0$, $y^{(1)}(x) + M(x)y(x-\Delta(x)) = 0$, $M(x) \geqq 0$, $\Delta(x) \geqq 0$, $A \leqq x < B$, $-\infty < A < B \leqq \infty$, essentially under the hypotheses and in the general setting of the author's work [same journal (N.S.) **4**, no. 5(33), 99–141 (1949); these Rev. **11**, 365]. Under suitable initial conditions (with M, Δ, φ continuous) a unique solution $y(x)$ exists.

W. J. Trjitzinsky (Urbana, Ill.).

Dantinne, Nelly. Application de la méthode des approximations successives à l'intégration d'une équation différentielle aux différences. III. Étude de l'équation linéaire. Bull. Soc. Roy. Sci. Liège **19**, 119–130 (1950).

The author continues her earlier work [same Bull. **18**, 363–374, 445–461 (1949); these Rev. **11**, 726], discussing the linear equation in the present paper. The use of vector-matrix notation would greatly simplify the presentation.

R. Bellman (Stanford University, Calif.).

Wright, E. M. The stability of solutions of non-linear difference-differential equations. Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 18–26 (1950).

Consider the nonlinear differential-difference equation $\Delta(y) = \sum_{m,n} a_{mn} y^{(n)}(x+b_m) = g(x, y)$, $0 \leqq m \leqq M$, $0 \leqq n \leqq N$, where $g(x, w)$ is a function of $x, w_{1,m}, \cdots, w_{N-1,m}$, $0 \leqq m \leqq M$, vanishing to order higher than the first in the neighborhood of $w_{i,j} = 0$, and $g(x, y)$ results from setting $w_{i,j} = y^{(i)}(x+b_j)$ (the condition imposed by the author is actually weaker than this, but more difficult to describe). It is shown by a method depending upon the Laplace transform, which is distinct from one used previously [cf. Quart. J. Math., Oxford Ser. (1) **20**, 155–165 (1949); these Rev. **11**, 364], that if the real parts of the roots of the exponential polynomial $T(s) = \sum_{m,n} a_{mn} s^n e^{b_m s}$ are all bounded by $-d < 0$, then for any $\epsilon > 0$, a $\delta = \delta(\epsilon, c)$ may be found such that $|y^{(n)}(x)| \leqq \delta$, $0 \leqq x \leqq b_m$, implies $|y^{(n)}(x)| \leqq \epsilon e^{-cx}$ for all $x \geqq 0$, $0 \leqq n \leqq N$, $0 < c < d$.

R. Bellman.

Hayes, N. D. Roots of the transcendental equation associated with a certain difference-differential equation. J. London Math. Soc. **25**, 226–232 (1950).

The following result is demonstrated. "The roots of $se^s - ae^s - b = 0$ lie to the left of $\Re(s) = 0$, if and only if $a < 1$, and $-a < -b < (v^2 + a^2)^{\frac{1}{2}}$, where v is the root of $v \cot v = a$ such that $0 < v < \frac{1}{2}\pi$." This result is useful in connection with the stability theory of the nonlinear differential-difference equation $y'(x+1) = ay(x+1) + by(x) + f(x, y, y(x+1))$, investigated by the reviewer [Ann. of Math. (2) **50**, 347–355 (1949); these Rev. **10**, 715], and, as a special case of more general functional equations, investigated by Wright [Quart. J. Math., Oxford Ser. (1) **20**, 155–165 (1949); these Rev. **11**, 364].

R. Bellman (Stanford University, Calif.).

de Bruijn, N. G. On some linear functional equations. Publ. Math. Debrecen **1**, 129–134 (1950).

In continuation of previous work [cf. Amer. J. Math. **71**, 313–330 (1949); these Rev. **10**, 541] the author treats the equation $w(x)f'(x) + f(x) - f(x-1) = 0$. It is shown that if $w(x)$ is not too small as $x \to \infty$, then every solution tends to a constant $f(\infty)$, and an estimate is obtained for $|f(x) - f(\infty)|$. A sufficient condition is $w(x) \geqq 1/\log x$ as $x \to \infty$. Several examples are given.

R. Bellman.

Germay, R. H. Sur les équations récurrentes définissant des fonctions implicites. Bull. Soc. Roy. Sci. Liège **19**, 74–82 (1950).

Following a method of successive approximations used by Goursat to establish the existence of implicit functions, the author considers the solution of the recurrent system $z_n = z_0 + f(x, y, z_n, z_{n+1})$, $n = 1, 2, \cdots$, under certain restrictions upon the f_n. These results are analogous to those obtained by Bruwier for recurrent differential systems [Mém. Soc. Roy. Sci. Liége (3) **17**, no. 4, 1–48 (1932)].

R. Bellman (Stanford University, Calif.).

Bagchi, Haridas, and Chatterji, Phatik Chand. Note on a functional equation, connected with the Weierstrassian function $\wp(z)$. Bull. Calcutta Math. Soc. **42**, 49–52 (1950).

The authors state that the only nonconstant functions $f(z)$, analytic except for poles in the finite plane, that satisfy the functional equation

$$\begin{vmatrix} f(x) & f(y) & f(x+y) \\ f'(x) & f'(y) & -f'(x+y) \\ 1 & 1 & 1 \end{vmatrix} = 0$$

are $f(z) = \wp(\lambda z)$, where λ and the constants g_2, g_3 of $\wp(z)$ are arbitrary. This is slightly in error, since whenever $f(z)$ is a solution so are $f(z) + C$ and $Cf(z)$, where C is an arbitrary constant. The authors' proof misses this fact because at one point $f(z)$ is replaced by its principal part at the pole $z = 0$, ignoring the other terms in the Laurent expansion for $f(z)$.

I. M. Sheffer (State College, Pa.).

Integral Equations

Harazov, D. F. On the distribution of the characteristic values of integral equations with kernel rational in the parameter. Doklady Akad. Nauk SSSR (N.S.) **71**, 1033–1035 (1950). (Russian)

The author studies the equation

$$(1) \quad u(x) = \int_a^b \Big[G_0(x, y) + \lambda \sum_{j=1}^{p} \sum_{k=1}^{m_j} (\lambda - \lambda_j)^{-k} G_k^{(j)}(x, y)\Big] u(y)\, dy,$$

G_0, $G_k^{(j)}$ real, symmetric, and of summable square in S ($a \leqq x, y \leqq b$), λ_j real, λ a complex parameter. A study is made of the distribution of the characteristic values of (1) in the plane of λ; sufficient conditions are found for the existence of at least one such value. It is assumed that $\int_a^b \varphi^2 dx - \int_a^b \int_a^b G_0(x, y)\varphi(x)\varphi(y)\, dx dy > 0$ whenever $\int_a^b \varphi^2 dx \neq 0$ exists. Let $m = \max(m_1, \cdots, m_p)$, $\lambda_r = \max(\lambda_1, \cdots, \lambda_p)$. Some of the results are as follows. If $G_k^{(j)}$ are semidefinite negative, then there are no nonreal characteristic values of (1), for which $0 < \arg(\lambda - \lambda_r) \leqq \pi/m$, $2\pi - \pi/m \leqq \arg(\lambda - \lambda_r) < 2\pi$. An equation

$$u(x) = \int_a^b \Big[G_0(x, y) + \lambda \sum_{j=1}^{p} (\lambda - \lambda_j)^{-1} G^{(j)}(x, y)\Big] u(y)\, dy,$$

where $G^{(j)}(x, y)$ are semidefinite negative kernels, can have only real characteristic values. If $G_k^{(j)}(x, y)$ are semidefinite negative kernels and among the λ_j there is a negative

number, or (in the contrary case) among the m_j there is an odd integer, then (1) has a real characteristic value.
W. J. Trjitzinsky (Urbana, Ill.).

Weinberg, M. Existence of eigen-functions for a class of non-linear integral equations. Učenye Zapiski Moskov. Gos. Univ. **100**, Matematika, Tom I, 85–92 (1946). (Russian. English summary)

Under consideration is the nonlinear integral equation (1) $\lambda u(x)=\int_B K(x,y)g(u(y),y)dy$, where the kernel $K(x,y)$ is bounded, symmetric, positive, with admissible discontinuities, and $g(u,x)$ is continuous, $g(0,x)\equiv 0$, $ug(u,x)\geqq 0$, $|g(u,x)|\geqq|f(u)|$, where $f(u)$ is nondecreasing. It is shown that (1) has a solution $u(x)=\psi(x)\not\equiv 0$, which is a characteristic function corresponding to a real characteristic value $\lambda\neq 0$. The developments of the author are connected with some of those of Lichtenstein [Vorlesungen über einige Klassen nicht-linearer Integralgleichungen . . . , Springer, Berlin, 1931] and Golomb [Math. Z. **39**, 45–75 (1933)].
W. J. Trjitzinsky (Urbana, Ill.).

Weinberg, M. On Hammerstein's theorem for non-linear integral equations. Učenye Zapiski Moskov. Gos. Univ. **100**, Matematika, Tom I, 93–103 (1946). (Russian. English summary)

It is shown that, if $u(x)=v(\lambda_0,x)$ is a solution of $u(x)=\lambda\int_B K(x,y)f(u(y),y)dy$ (the kernel $K(x,y)$ continuous, symmetric, positive; $f(x,y)$ continuous for x in B; $f_u(u,x)$ continuous for x in B, when u is on a certain interval) for $\lambda=\lambda_0$, while λ_0 is not a characteristic number of the equation $\varphi(x)=\lambda\int_B K(x,y)(\partial/\partial u)f(v(\lambda_0,y),y)\varphi(y)dy$, then there exists an interval I, containing λ_0, so that for every λ in I there is a solution $u(x)=v(\lambda,x)$ such that $v(\lambda,x)\rightrightarrows v(\lambda_0,x)$ (as $\lambda\to\lambda_0$). This theorem is similar to a result of Hammerstein [Acta Math. **54**, 117–176 (1930)], but differs from the latter both as regards the conditions involved and the method of proof. *W. J. Trjitzinsky* (Urbana, Ill.).

Bykov, Ya. V. On the problem of singular functions of non-linear integral equations. Doklady Akad. Nauk SSSR. (N.S.) **72**, 449–452 (1950). (Russian)

We consider the equation (1) $\lambda\varphi(x)=$

$$\int_B\cdots\int_B K(x,v)f(v,t_1,\cdots,t_m,\varphi(t_1),\cdots,\varphi(t_m),\varphi(v))d\sigma_m dv,$$

where B is a closed bounded region in r-space and

$$f(v,t_1,\cdots,t_m,0,\cdots,0)\equiv 0.$$

Condition (A): $K(x,v)$ is a bounded symmetric positive kernel with admissible discontinuities; $f(v,t_1,\cdots,x_1,\cdots,y)$ in a neighborhood of $x_1=\cdots=x_m=y=0$ is (i) measurable in $(v,t_1,\cdots,t_m)$ in B, and

$$|f(v,t_1,\cdots,x_1,\cdots,y)|\leqq F(v,t_1,\cdots)\varepsilon L_{2,m+1},$$

(ii) continuous in $(x_1,\cdots,y)$ and has, with respect to $x_1,\cdots,x_m$, continuous partials, while

$$\frac{\partial f(v,t_1,\cdots,x_1,\cdots,y)}{\partial x_j}=\frac{\partial f(t_j,t_1,\cdots,t_{j-1},v,t_{j+1},\cdots,t_m,x_1,\cdots,x_{j-1},y,x_{j+1},\cdots,x_m,x_j)}{\partial x_j},$$

(iii) $|\partial f/\partial x_j|\leqq F_1(v,s_1,\cdots,s_m)\varepsilon L_{2,m+1}$. It is proved that under condition (A) equation (1) has at least a denumerable infinity of characteristic functions; a similar result is established for the equation

$$\lambda\varphi(x)=\sum_0^\infty\int_B\cdots\int_B K(x,v)\times f_m(v,s_1,\cdots,s_m,\varphi(s_1),\cdots,\varphi(s_m),\varphi(v))d\sigma_m dv,$$

when a certain condition, termed (B), and resembling condition (A), is satisfied. *W. J. Trjitzinsky* (Urbana, Ill.).

Mihlin, S. G. Singular integral equations. Amer. Math. Soc. Translation no. **24**, 116 pp. (1950).

Translated from Uspehi Matem. Nauk (N.S.) **3**, no. 3(25), 29–112 (1948); these Rev. **10**, 305.

Parodi, M. Sur une propriété de l'équation intégrale

$$\int_0^\infty\frac{t^x f(x)dx}{\Gamma(1+x)}=g(t).$$

Ann. Soc. Sci. Bruxelles. Sér. I. **62**, 24–26 (1948).

If $\varphi(p)$ and $\theta(p)$ are the operational images respectively of $f(t)$ and $g(t)$, the integral equation of the title becomes $\varphi(\log p)=\theta(p)\log p$ and hence can be solved. The author discusses a slightly more general equation and applies his results to the function $\nu(t,n)$ introduced by Colombo [C. R. Acad. Sci. Paris **216**, 368–369 (1943); these Rev. **5**, 238].
A. Erdélyi (Pasadena, Calif.).

Jaeckel, K. Ermittlung einer Reihendarstellung des Kernes $\ln r$ in elliptischen Koordinaten. Z. Angew. Math. Mech. **30**, 184–187 (1950).

Let $z=x+iy$, $\zeta=\xi+i\eta$, $w=u-iv$, $W=U-iV$. Making the transformation $\zeta=-\cosh w$, $z=-\cosh W$, the function $(\zeta-z)^{-1}$ may be expanded in a Fourier series in $\cos nv$ and $\sin nv$. The author determines the coefficients and finds, after integrating,

$$\log(\zeta-z)=W+\text{constant}-2\sum_1^\infty n^{-1}e^{-nW}\cosh nw$$

for $U>u$, and

$$\log(\zeta-z)=\text{constant}-2\sum_1^\infty n^{-1}e^{-nw}\cosh nW$$

for $U<u$. Taking real and imaginary parts, the results can be immediately applied to obtain solutions of the integral equation $\int_0^{2\pi}\varphi(v)\log r(U,u;V,v)dv=\lambda(U,u)\varphi(V)$, where $r=|\zeta-z|$, as well as of various other related integro-differential equations. *J. V. Wehausen* (Providence, R. I.).

Functional Analysis, Ergodic Theory

Mikusiński, Jan G.-. Sur certains espaces abstraits. Fund. Math. **36**, 125–130 (1949).

Let A be an additive Abelian group and let $A^*\subset A$ be the set of "nonnegative elements" of A satisfying (1) $0\varepsilon A^*$, (2) $a, b\varepsilon A^*$ imply $a+b\varepsilon A^*$, (3) if $a\varepsilon A^*$, $b\varepsilon A$ imply $b-\lambda a\varepsilon A^*$ for any positive integer λ, then $a=0$. Then A^* defines a partial order on A and an absolute value function $|a|$ from A to A^* with the usual properties. A sequence $\{a_n\}$ of A is said to converge if there exists $c\varepsilon A^*$ such that if λ is any positive integer $\lambda|a_m-a_n|\leqq c$ for m,n sufficiently large. Then A_l is defined to be a linear space with the above properties. It is shown that certain classes of functions form spaces of type A and A_l under the natural partial ordering,

and that the above definition of convergence is equivalent to well-known types of convergence in these classes. For example, convergence almost everywhere of sequences of measurable functions and convergence on compact sets for sequences of continuous functions can be characterized in this manner. *R. E. Fullerton* (Madison, Wis.).

Bielecki, Adam. Sur certaines inégalités dans les espaces abstraits de J. G.-Mikusiński. Fund. Math. 36, 131–132 (1949).

If A, A_l are spaces of the type defined in the preceding review, the inequality $||a|-|b||\leqq|a\pm b|$ is true in A_l but not necessarily in A. In both A and A_l the inequalities $||a|-|b||\leqq 3|a\pm b|$ and $|2|a|-2|b||\leqq 2|a\pm b|$ hold. These results imply (1) a, $a_\nu \varepsilon A$ and $\lim_{\nu\to\infty} a_\nu = a$, $\nu = 1, 2, \cdots$ imply $\lim_{\nu\to\infty} |a_\nu| = |a|$; (2) if A is complete A^* is complete. *R. E. Fullerton* (Madison, Wis.).

Bielecki, Adam. Sur quelques conditions nécessaires et suffisantes pour que l'espace A_l de J. G.-Mikusiński soit topologique au sens de Kuratowski. Fund. Math. 36, 133–136 (1949).

Let A_l be defined as in the second preceding review. It is shown that the limiting process defined on A generates a Kuratowski topology if any of the following equivalent conditions hold. (1) If a, a^i, $a_n{}^i \varepsilon A$, $i, n = 1, 2, 3, \cdots$, and if $\lim_{i\to\infty} a^i = a$, $\lim_{n\to\infty} a_n{}^i = a^i$, there exists a subsequence $a_{n(\lambda)}^{i(\lambda)}$ of the $a_n{}^i$ such that $\lim_{\lambda\to\infty} a_{n(\lambda)}^{i(\lambda)} = a$. (2) If $b_n \varepsilon A^*, n = 1, 2, 3, \cdots$, there exists a sequence of positive numbers $\{\beta_n\}$ and an element $b \varepsilon A^*$ with $\beta_n \cdot b_n \leqq b$, $n = 1, 2, 3, \cdots$. (3) If $b_n \varepsilon A^*$, $n = 1, 2, \cdots$, there exists a sequence of positive numbers σ_n with $\lim_{n\to\infty} \sigma_n b_n = 0$. The author shows that the class of all real valued functions finite on $[0, 1]$ is an A_l-space which is not topological. *R. E. Fullerton* (Madison, Wis.).

Revuz, André. Sur une représentation canonique de certaines fonctionnelles croissantes. C. R. Acad. Sci. Paris 231, 22–24 (1950).

The author considers a compact space C, an ordered metric space E, and certain subspaces X of continuous mappings of C into E. A number of conditions are imposed on E relating the segments of E to the metric. He assumes X to be compact and bounded in its natural metric topology. Further conditions on the natural order in X and its relation to the topology of X are also assumed, in particular, X is a lattice. In this framework one can consider functionals $F(x)$ defined on X which are positive, nondecreasing, and continuous on the right. The author gives a necessary and sufficient condition that there be a positive Radon measure defined on X so that $F(x)$, for each x in X, is the measure of the set $E[y; y \leqq x]$. The condition is that if $S = \{u_1, \cdots, u_p, x\}$ is any finite system of elements in X where $u_i \leqq x$, $i = 1, \cdots, p$, then

$$\Delta(S) = F(x) - \sum_i F(u_i) + \cdots + (-1)^k \sum_{(i_1 \cdots i_k)} F[\inf u_{i_1} \cdots u_{i_k}] + \cdots + (-1)^p F[\inf u_1 \cdots u_p] \geqq 0.$$

B. Yood (Ithaca, N. Y.).

Mišík, L. Sur la propriété définie, d'après M. J. Novák, dans les espaces de fonctions continues. Acta Fac. Nat. Univ. Carol., Prague no. 187 (1948), 26–30 (1948). (Czech and French)

A sequence contained in a double sequence $x_{m,n}$, $m, n = 1, 2, \cdots$, is said to be a diagonal sequence if it contains only a finite number of elements of each row of $x_{m,n}$ (considered as a matrix). A point x of a space L provided with a convergence notion is said to have property ρ if there exists a double sequence $x_{m,n}$ of points of L such that each row converges to x but no diagonal sequence converges to x; $x_{m,n}$ is called a system ρ for x. The author relates the existence of a point with property ρ in a space $L(P)$ of continuous functions over a topological space P to standard convergence notions in $L(P)$. He characterizes systems ρ for the zero element in $L(P)$ when the convergence notion in $L(P)$ is pointwise convergence over P, and shows how to obtain such systems if P is normal. Proofs are not given. *S. B. Myers* (Ann Arbor, Mich.).

Mazurkiewicz, Stefan. Sur les espaces de variables aléatoires. Fund. Math. 36, 288–302 (1949).

A family $M = \{x\}$ of random variables x is called an EVA (espace de variables aléatoires) if, for any finite number of elements $x_1, \cdots, x_n$ from M, their n-dimensional joint distribution $F_{x_1,\cdots,x_n}(t_1, \cdots, t_n) = \Pr\{x_k \leqq t_k,\ k = 1, \cdots, n\}$ is given. The author proves that the completion M^* of an EVA M with respect to the Fréchet metric

$$\rho(x_1, x_2) = \inf_{\lambda>0} (\lambda + \Pr\{|x_1 - x_2| \geqq \lambda\})$$

is again an EVA, and shows that the universal EVA M consisting of all real-valued functions $x(\tau)$ defined on the unit interval $\{\tau | 0 \leqq \tau < 1\}$ can be considered as the completion $M_0{}^*$ of a subfamily M_0 of M consisting of all step functions $x(\tau)$ with the property that there exists an integer n such that $x(\tau)$ is constant in each of the intervals

$$\{k2^{-n} \leqq \tau < (k+1)2^{-n}\},$$

$k = 1, \cdots, 2^n$. *S. Kakutani* (New Haven, Conn.).

Ganapathy Iyer, V. On the space of integral functions. II. Quart. J. Math., Oxford Ser. (2) 1, 86–96 (1950).

The author continues his study of the set Γ of all entire functions, regarded as a linear space under the (metric) topology of uniform convergence on compact sets [J. Indian Math. Soc. (N.S.) 12, 13–30 (1948); these Rev. 10, 380]. The extension theorem for functionals is proved by means of the norms $|\alpha; R| = \sum |a_n| R^n$ where $\alpha(z) = \sum a_n z^n$, which serve to generate the topology of Γ. Some remarks are made on the nature and existence of bases and biorthogonal systems. It is observed that the sequence α^n is not a base for Γ unless α is linear. *R. C. Buck* (Madison, Wis.).

* **Nyman, Bertil. On the One-Dimensional Translation Group and Semi-Group in Certain Function Spaces.** Thesis, University of Uppsala, 1950. 55 pp.

A Beurling weight-function $p(x)$ $(-\infty < x < \infty)$ is characterized by the conditions $p(x) \geqq p(0) = 1$, $p(x+y) \leqq p(x)p(y)$, $p(\rho x) \geqq p(x)$ for $\rho > 1$. For such a $p(x)$, $\alpha = \lim_{x\to\infty} x^{-1} p(x)$ and $\beta = \lim_{x\to\infty} x^{-1} p(-x)$ exist and are $\geqq 0$; $p(x)$ is said to be of analytic type if $\alpha + \beta > 0$, of quasi-analytic type if $\alpha = \beta = 0$ and $\int_{-\infty}^{\infty} (1+x^2)^{-1} \log p(x)dx = \infty$, and of nonquasi-analytic type if this integral converges. Let $L_p{}^\infty$ be the space of measurable functions $\varphi(x)$ on $(-\infty, \infty)$ with norm $\sup |\varphi(x)|/p(x) < \infty$, and $L_p{}^1$ be the space of measurable functions $k(x)$ on $(-\infty, \infty)$ with norm $\int_{-\infty}^{\infty} |k(x)| p(x)dx < \infty$. The weak closure $\bar{D}$ of a set $D \subset L_p{}^\infty$ is defined as the set of functions $\varphi(x) \varepsilon L_p{}^\infty$ such that $\int_{-\infty}^{\infty} k(x)\psi(x)dx = 0$ for all $\psi \varepsilon D$, where $k(x) \varepsilon L_p{}^1$, implies $\int_{-\infty}^{\infty} k(x)\varphi(x)dx = 0$. The spectral set Λ_φ of a function $\varphi(x) \varepsilon L_p{}^\infty$ is defined as the set of all complex numbers s such that $e^{-sx} \varepsilon C_\varphi = \bar{D}$, where D is the set of all finite linear combinations of translations of $\varphi(x)$. Obviously, Λ_φ lies in the strip $-\alpha \leqq \sigma \leqq \beta$.

For $p(x)\equiv 1$, L_p^∞ becomes $L^\infty(-\infty,\infty)$, the space of bounded measurable functions on $(-\infty,\infty)$. A general "extinction" theorem of R. Godement for bounded functions on locally compact Abelian topological groups [Ann. Sci. École Norm. Sup. (3) 64 (1947), 119–138 (1948); these Rev. 9, 327] implies that if $\varphi(x)\in L^\infty(-\infty,\infty)$, $\varphi(x)\not\equiv 0$, then Λ_φ is not empty (that is, it contains a number it, t real). Earlier, A. Beurling [Acta Math. 77, 127–136 (1945); these Rev. 7, 61] had proved a similar extinction for a subspace of $L^\infty(-\infty,\infty)$, using, however, a topology ("narrow" topology) different from the above weak topology. In a paper which has not been published yet, Beurling proves that both extinction theorems remain valid in L_p^∞ for general $p(x)$ of nonquasi-analytic type. Moreover, the spectral sets defined by the weak, and by the narrow topology, respectively, are identical. The author proves by examples that there are weight-functions $p(x)$ of analytic and of quasi-analytic type, for which L_p^∞ contains functions $\varphi(x)\not\equiv 0$ which have empty Λ_φ.

In the special case $p(x)=1$ $(x\geqq 0)$, $p(x)=\infty$ $(x<0)$, L_p^∞ reduces to $L^\infty(0,\infty)$, the space of functions defined and essentially bounded for $x\geqq 0$, L_p^1 becomes $L^1(0,\infty)$, and only the translations $\varphi(x+\tau)$ with $\tau\geqq 0$ of a function $\varphi(x)\in L^\infty(0,\infty)$ are considered. The author proves that Λ_φ is never empty, provided that $\varphi(x)$ is not ~ 0 outside a finite interval (extinction theorem for $L^\infty(0,\infty)$). The proof depends on a study of the Laplace transform $\Phi(s)=\int_0^\infty \varphi(x)e^{sx}dx$ of $\varphi(x)$. By a method of analytic continuation used by T. Carleman [L'Intégrale de Fourier . . . , Publications Scientifiques de l'Institut Mittag-Leffler, 1, Uppsala, 1944; these Rev. 7, 248] the author shows that, if $\varphi(x)$ is mean-periodic (that is, $C_\varphi\not\equiv L^\infty(0,\infty)$), then $\Phi(s)$ can be continued analytically into $\sigma>0$ by $\Psi(s)/K(s)$. Here $K(s)=\int_0^\infty k(x)e^{-sx}dx$, where $k(x)$ is any function in $L^1(0,\infty)$ which is $\not\equiv 0$ and satisfies the condition that $\psi(y)=\int_0^\infty k(x)\varphi(x+y)dx=0$ for $y\geqq 0$. Finally, $\Psi(s)=\int_{-\infty}^0 \psi(x)e^{sx}dx$. It follows that $\Phi(s)$ is single-valued, and regular in the entire plane with the possible exception of a null-set of singularities on the imaginary axis, and poles $\{s_n\}$ in $\sigma>0$ satisfying $\sum \Re(s_n)/(1+|s_n|^2)<\infty$. The set of singularities of $\Phi(s)$ is identical with Λ_φ. (In the proof of this statement the following unpublished result of Beurling is used. Let D be a linear subset, invariant under translation, of the space of bounded continuous functions on $(-\infty,\infty)$. Let $\bar{D}$ be the weak closure of D. Then $\bar{D}=D^*$, where D^* is the closure of D in a topology defined as follows: $f(x)\in D^*$ if, for every positive continuous function $\omega(x)$ with $\lim_{x\to+\infty}\omega(x)=\infty$, there exists a function $g(x)\in D$ such that $|f(x)-g(x)|<\omega(x)$ for $-\infty<x<\infty$.) It is further shown that if $s\in\Lambda_\varphi$ has multiplicity q, that is, if $x^\nu e^{-sx}\in C_\varphi$ for $\nu=0,1,\cdots,q-1$, but not for $\nu=q$, then s is a pole of $\Phi(s)$ of order q. If Λ_φ lies entirely in $\sigma>0$, spectral synthesis is possible in the following sense: at least one translation $\varphi(x+A)$, $A\geqq 0$, of the mean-periodic function $\varphi(x)\in L^\infty(0,\infty)$ belongs to the weak closure of the linear manifold spanned by $\{x^\nu e^{-sx}\}$, $s\in\Lambda_\varphi$, $\nu=0,1,\cdots,q-1$ (q is the multiplicity of s).

The above results for $L^\infty(0,\infty)$ imply the following "Wiener closure theorem" for $L^1(0,\infty)$. Let $k(x)\in L^1(0,\infty)$, $k(x)=0$ for $x<0$. The set $\{k(x-\tau)\}$, $\tau\geqq 0$, is fundamental in $L^1(0,\infty)$ if and only if $K(s)=\int_0^\infty k(x)e^{-sx}dx\not\equiv 0$ in $\sigma\geqq 0$, and $k(x)$ not ~ 0 in any interval $(0,A)$, $A>0$. A similar result holds for families of functions $\{k(x)\}$ instead of $k(x)$. The author also considers the space L_p^1, where $p(x)=e^{\alpha x}$ $(x\geqq 0)$, $p(x)=e^{-\beta x}$ $(x<0)$, $\alpha\geqq 0$, $\beta\geqq 0$, $\alpha+\beta>0$. He finds that the set $\{k(x-\tau)\}$, $-\infty<\tau<\infty$, $k(x)\in L_p^1$, is fundamental in L_p^1 if and only if $K(s)=\int_{-\alpha}^\beta k(x)e^{-sx}dx\not\equiv 0$ in $-\alpha\leqq\sigma\leqq\beta$, and $\exp\{-\pi|t|/(\alpha+\beta)\}\log|K(\sigma+it)|\to 0$ as $t\to\pm\infty$ for $-\alpha<\sigma<\beta$.

If $F(s)$ is bounded in $\sigma>0$, or satisfies a Hardy condition: $\int_{-\infty}^\infty |F(\sigma+it)|^p dt<M$ $(\sigma>0)$ for some $p>0$, and if $F(s)$ is regular in $\sigma>0$ (in particular, if $F(s)$ is the Laplace transform $\int_0^\infty f(x)e^{-sx}dx$ of a function $f(x)\in L^1(0,\infty)$ or $L^2(0,\infty)$), then $F(s)$ can be uniquely factorized in the form $F(s)=F_0(s)F_1(s)$, where

$$F_0(s)=e^{-As}B(s)\exp\left\{-\int_{-\infty}^\infty [(1-iys)/(s-iy)]d\mu(y)\right\},$$

$$F_1(s)=c\exp\left\{\frac{1}{\pi}\int_{-\infty}^\infty \frac{1-iys}{s-iy}\,\frac{\log|F(iy)|}{1+y^2}dy\right\}.$$

Here $B(s)$ is the Blaschke product containing the zeros $\{s_n\}_1^\infty$ of $F(s)$ in $\sigma>0$, $A\geqq 0$, $|c|=1$, $\mu(y)$ a real non-decreasing bounded function, whose points of increase form a set of measure 0 [W. Kryloff, Rec. Math. [Mat. Sbornik] N.S. 6(48), 95–138 (1939); these Rev. 1, 308]. For $L^2(0,\infty)$, K. Karhunen [Ark. Mat. 1, 141–160 (1950); these Rev. 11, 607] made use of this decomposition theorem to prove that if $k(x)\in L^2(0,\infty)$, $k(x)=0$ for $x<0$, then the set $\{k(x-\tau)\}$, $\tau\geqq 0$, is fundamental in $L^2(0,\infty)$ if and only if the inner factor $K_0(s)$ of $K(s)=\int_0^\infty k(x)e^{-sx}dx$ $(\sigma>0)$ is $\equiv 1$. It follows that an extinction theorem analogous to the one proved for $L^\infty(0,\infty)$ does not hold in $L^2(0,\infty)$. The author also gives a more precise form to Karhunen's result: $k(x)\in L^2(0,\infty)$ will belong to the L^2 closure of the linear manifold spanned by $\{k(x-\tau)\}$, $\tau\geqq 0$, $k(x)\in L^2(0,\infty)$, if and only if the inner factor $K_0(s)$ of $K(s)$ is a divisor of the inner factor $H_0(s)$ of $H(s)=\int_0^\infty h(x)e^{-sx}dx$ (that is, if and only if $H_0(s)/K_0(s)$ is itself an inner factor $G_0(s)$). It is also proved that the L^2 closure of the linear manifold spanned by the translations $k_1(x-\tau)$, $k_2(x-\tau)$, $\tau\geqq 0$, of two functions in $L^2(0,\infty)$ is identical with the L^2 closure of the linear manifold spanned by the translations $k_3(x-\tau)$, $\tau\geqq 0$, of a suitably chosen function $k_3(x)$. This result is used to prove that the Riemann hypothesis is true if and only if the set $\{[\alpha/x]-\alpha[1/x]\}$, $0<\alpha\leqq 1$, is fundamental in $L^2(0,1)$.

A final chapter is devoted to extinction theorems for $l^\infty(0,\infty)$, the space of bounded sequences $A=\{a_n\}_0^\infty$, with norm $\|A\|=$least upper bound $|a_n|$. [For $l^\infty(-\infty,\infty)$, Godement's results will immediately give an extinction theorem.] Let $l^1(0,\infty)$ be the space of sequences $B=\{b_n\}_0^\infty$, with norm $\|B\|=\sum|b_n|<\infty$, T^kA the sequence $\{a_{n+k}\}_{n=0}^\infty$ $(k\geqq 0)$, and C_A the weak closure of the manifold spanned by the elements T^kA ($x\in C_A$ if and only if the equations $\sum_{n=0}^\infty a_{n+k}b_n=0$, $k=0,1,\cdots$, imply $\sum x_nb_n=0$ whenever $B\in l^1(0,\infty)$). Finally, the spectral set Λ_A of A is defined as the set of complex numbers λ such that $\{\lambda^n\}_0^\infty\in C_A$. Obviously, $\lambda\in\Lambda_A$ implies $|\lambda|\leqq 1$. The following extinction theorem holds. If $A\in l^\infty(0,\infty)$, $A\not\equiv 0$, then Λ_A is not empty. It is stated that this result may be proved by using the theory of normed rings, or by a study of the function $\sum a_nz^n$, defined in $|z|<1$. If A is mean-periodic (that is, $C_A\not\equiv l^\infty(0,\infty)$), $\sum a_nz^n$ can be continued analytically into $|z|>1$; the set of singularities of $f(1/z)$ is identical with Λ_A. A second extinction theorem for $l^\infty(0,\infty)$ is proved, using the following ("narrow") topology: $X_k=\{x_{k,n}\}_{n=0}^\infty$ converges narrowly to $X=\{x_n\}$ if $x_{k,n}\to x_n$ as $k\to\infty$, $n=0,1,\cdots$, and $\|X_k\|\to\|X\|$. Define $\bar{C}_A$ as the smallest linear, narrowly closed manifold, containing A and invariant under translation to the left. If $A\in l^\infty(0,\infty)$, and $A\not\equiv 0$, then $\bar{C}_A$ contains an element $\{\lambda^n\}_0^\infty$.

J. Korevaar (Lafayette, Ind.).

Hille, Einar. On the differentiability of semi-group operators. Acta Sci. Math. Szeged **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 19–24 (1950).

Suppose A is a closed distributive operator on the Banach space B to B with domain dense in B, whose resolvent $R(\lambda, A)$ satisfies $\|R(\lambda, A)\| \leqq \lambda^{-1}$ for positive λ. Then it is known that A is the infinitesimal generator of a semigroup $T(\xi|\xi>0)$ of linear operators on B to B. If T has a derivative $T'(\xi)$ then $(T'(\xi))x = AT(\xi)x$. The paper considers some order of growth results on $T(\xi)$ which are interrelated with hypotheses on A, or its range, or on $R(\lambda, A)$. These lead to conditions under which $T'(\xi)$ and higher derivatives exist.

D. G. Bourgin (Urbana, Ill.).

Kozlov, V. Ya. On a generalization of the concept of basis. Doklady Akad. Nauk SSSR (N.S.) **73**, 643–646 (1950). (Russian)

Let E be a Banach space and let $T=\{t_{ij}\}$ be a matrix defining a regular method of summability of sequences of real numbers; that is, (a) $\lim_{i\to\infty} t_{ij}=0$ for each j, (b) $\lim_{i\to\infty} \sum_j t_{ij}=1$, and (c) $\sup_i \sum_j |t_{ij}| < \infty$. A sequence $\{x_n\}$ of points of norm 1 in E is called a Banach basis if for each x in E there exist numbers $c_n(x)$, uniquely determined by x, such that the partial sums $S_k(x) = \sum_{i\leqq k} c_i(x)x_i$ converge (in the norm topology) to x. Also, $\{x_n\}$ is called a T-basis if the sequence $S_k(x)$ is summable to x by the method T, i.e., if $\sum_j t_{ij}S_j(x)$ converges to x for each x in E. For example, if T is the matrix of Cesàro summability, Fejér's theorem asserts that suitable trigonometric functions form a T-basis for the continuous functions, although it is known that these same functions are not a Banach basis. Theorem 1. Let T satisfy the above conditions and also suppose $t_{ij}=0$ if $j>i$. If $\{x_n\}$ is a T-basis in Hilbert space H, then there exist linear functionals F_i such that $F_i(x_j)=\delta_{ij}$ for $i, j=1, 2, \cdots$.

In E two Banach bases are called equivalent if

$$F_n(x)x_n = \Phi_n(y)y_n$$

for every integer n and x in E, where $\{F_n\}$ and $\{\Phi_n\}$ are the sequences of coefficient functionals $\{c_n\}$ for $\{x_n\}$ and $\{y_n\}$ respectively. If R is the set of equivalence classes of Banach bases in E, define a distance

$$\rho(X, Y) = \sup_{\|x\|\leqq 1} \sup_N \Big\| \sum_{i\leqq N} F_i(x)x_i - \sum_{i\leqq N} \Phi_i(x)y_i \Big\|,$$

where $\{x_n\}\in X$ and $\{y_n\}\in Y$. Theorem 2. If E is a Banach space with a Banach basis, then R, metrized in this way, is a complete metric space. [Reviewer's remark. The usual proof for Banach bases, that the $c_i(x)$ have the properties desired for $F_i(x)$, can be used to prove theorem 1 without the extra hypotheses on T and E.]

M. M. Day.

Morse, Marston, and Transue, William. Functionals F bilinear over the product $A\times B$ of two pseudo-normed vector spaces. II. Admissible spaces A. Ann. of Math. (2) **51**, 576–614 (1950).

This paper is the second one in a series concerned with the general theory indicated by the title [for the first paper see the same Ann. (2) **50**, 777–815 (1949); these Rev. **11**, 185]. For terminology the reader is referred to the review of the earlier paper. The object of the present paper is to verify that certain significant special spaces satisfy the conditions stated in the first paper, and therefore the general representation theory for bilinear functionals, developed in the first paper, is applicable to the product $A\times B$ of two such spaces. The definition of these spaces involves a base-set E, located on the real number-line R_1, and satisfying certain restrictions appropriate to the setting of the theory. In terms of E a class of real-valued functions $x(s)$ belonging to E is defined, and the space $A=A(E)$ is the vector space of these functions, with an appropriate pseudo-norm being assigned. The following important cases are studied in detail: (i) the space $C(E_j)$, with the pseudo-norm $\max |x(s)|$, $s\in R_1$; (ii) the space $L_p(E)$, $p>1$, with the pseudo-norm $[\int_{R_1} |x(s)|^p ds]^{1/p}$; (iii) the space $L(E)$, corresponding to the case $p=1$; (iv) the spaces $\Psi^*(E)$, $\Phi(E)$, defined in terms of a generalized Hellinger integral. The verification of conditions I–V of the general representation theory in these various cases depends upon a number of ingenious estimates which may be of independent interest in their own right. The generality of the results obtained is illustrated by remarks showing that the spaces considered previously by Orlicz [Bull. Int. Acad. Polon. Sci. Cl. Sci. Math. Nat. Ser. A. Sci. Math. no. 8/9, 207–220 (1932)] are special instances of the spaces $\Phi(E)$.

T. Radó (Columbus, Ohio).

McShane, E. J. Linear functionals on certain Banach spaces. Proc. Amer. Math. Soc. **1**, 402–408 (1950).

The purpose of this paper is to obtain a representation for linear functionals on spaces L_p ($p>1$) and some generalizations of such spaces. The proof applies to spaces L_p over arbitrary sets X in a family of whose subsets a measure function is defined; no decomposition of X is involved. Instead of making use of the Radon-Nikodým theorem, the author bases his proof on the uniform convexity property of L_p. The theorem is extended to spaces $L_p(B)$ where B is a uniformly convex Banach space such that for every $g, f\in B$, $d\|g+tf\|/dt$ exists. The existence of this derivative means that the plane of support at each point on the unit sphere of B is unique. The paper concludes with a proof of the fact that $L_p(B)$ is uniformly convex if B is uniformly convex. It should be mentioned that R. C. James [Trans. Amer. Math. Soc. **61**, 265–292 (1947), theorem 8.2; these Rev. **9**, 42] obtained results very similar to those of the author.

R. S. Phillips (Princeton, N. J.).

Citlanadze, È. S. Some problems of nonlinear operators and the calculus of variations in spaces of Banach type. Uspehi Matem. Nauk (N.S.) **5**, no. 4(38), 141–142 (1950). (Russian)

This paper consists of definitions and theorems with no proofs. Let F be a functional on a Banach space E; its differential $dF=(dF(x; h)|x\in E, h\in E)$ is for each fixed $x\in E$ a linear functional L_Fx. If E is reflexive and has a basis, and $|F(x+h)-F(x)-dF(x; h)| \leqq C_F\|h\|^2$ for some constant C_F, then L_F is completely continuous on E to its adjoint $\bar{E}$, and if F is weakly continuous, weak convergence of $x_1, x_2, \cdots$ in E implies strong convergence of $L_Fx_1, L_Fx_2, \cdots$ in $\bar{E}$. Let φ be a differentiable functional on E, a a constant, the set u of x such that $\varphi(x)=a$, bounded, $E_\varphi x$ the linear functional whose value at h is $d\varphi(x; h)$. Assume that $d\varphi(x; x)\geqq C_1>0$ on u, and that L_F and E_φ are Lipschitzian on E to $\bar{E}$, $[V]$ is a homotopy class of compact sets on u, and c is the supremum of $\min\{F(x)|x\in V\}$ for all $V\in[V]$. An element $x\in u$ is a "characteristic element" if $L_Fx=\lambda E_\varphi x$ for some real λ; this λ is a "characteristic number." If L_F and E are as above and L_F is completely continuous, and E_φ has a continuous inverse, then either there is a characteristic element x of u at which $F(x)=c$, or there is a sequence $x_1, x_2, \cdots$ of points of u such that $\|L_Fx_n\|\to 0$, $F(x_n)\to c$. The final theorem is: Let (1) E be reflexive, with

basis, (2) $\varphi(x)=\|x\|$ be differentiable, (3) E_φ Lipschitzian in S_1, having continuous inverse, (4) $F(-x)=F(x)$, $F(x)>0$ on S_1 (S_1 the surface of the sphere S_1), $F(\theta_E)=\theta$ and $F(x)\neq 0$ elsewhere, (5) L_F completely continuous, Lipschitzian in S_1, and (6) $L_F(\theta_E)=0$, $\|L_F(x)\|>0$ elsewhere. Then there exists an infinite system of geometrically distinct normed characteristic elements x_n, $L_F x_n=\lambda_n E_\varphi x_n$, $\|x_n\|=1$, for which the critical values $F(x_n)\to 0$ as $n\to\infty$. [Presumably the author meant to exclude a neighborhood of θ_E in (2), (3), (5), or perhaps to restrict x to S_1; as the conditions stand, they cannot be satisfied.] It is stated that the theorems are applicable to the theory of nonlinear integral equations and infinite algebraic systems in infinitely many unknowns.

E. J. McShane (Charlottesville, Va.).

Krasnosel'skiĭ, M. A. On a fixed point principle for completely continuous operators in functional spaces. Doklady Akad. Nauk SSSR (N.S.) **73**, 13–15 (1950). (Russian)

Let F be a completely continuous, possibly nonlinear, operator in a Banach space E, and define a ("completely continuous") vector field in E by $\Phi x=Fx-x$. The following generalization of the Schauder-Leray fixed point theorem is stated. In a solid sphere T of E let a completely continuous vector field Φ be given such that on the surface S of T the vector $\Phi x\neq 0$ and $\Phi x^*/\|\Phi x^*\|\neq \Phi x/\|\Phi x\|$, where x and x^* are diametral points of S. Then there is at least one solution in T of $\Phi y=0$. Application is made to integral equations of the form

$$x(s)=\lambda\sum_{n=0}^{\infty}\int_0^1\cdots\int_0^1 K_{2n+1}(s;s_1,\cdots,s_{2n+1})\times x(s_1)\cdots x(s_{2n+1})ds_1\cdots ds_{2n+1}+Ax,$$

where A is completely continuous and $\|Ax\|$ is sufficiently small for $\|x\|=1$. The author next considers operators in L^2 of the form $\mathfrak{f}x=f(x(s),s)$ where $f(u,s)$ is defined for $-\infty<u<\infty$ and $s\in G$, G a set of finite measure. A necessary and sufficient condition is stated for $\mathfrak{f}$ to be continuous in L^2, and it is pointed out that $\mathfrak{f}$ has certain properties of linear operators. The results are applied to establish an existence theorem under certain restrictions for the Hammerstein integral equation $x(s)=\int_G K(s,t)f(x(t),t)dt$. No proofs are given. *J. V. Wehausen* (Providence, R. I.).

Vaĭnberg, M. M. On the continuity of some operators of special type. Doklady Akad. Nauk SSSR (N.S.) **73**, 253–255 (1950). (Russian)

Let $f(u,x)$ be defined for u real and for $x\in B$, a measurable subset of Euclidean n-space, and assume f is continuous for x fixed, and measurable for u fixed. If $u(x)$ is defined on B, let $\mathfrak{h}u(x)=f(u(x),x)$; $\mathfrak{h}u$ is measurable if u is. Fix $p\geqq 1$ and $p_1\geqq 1$; if $u\in L_p$ and $\mathfrak{h}u\in L_{p_1}$, for F measurable, $F\subset B$, define $\|u\|_F=(\int_F|u(x)|^p dx)^{1/p}$ and $\|\mathfrak{h}u\|_F=(\int_F|\mathfrak{h}u(x)|^{p_1}dx)^{1/p_1}$; $\mathfrak{h}$ is called continuous from L_p to L_{p_1} if and only if, for every $v\in L_p$, $\mathfrak{h}v\in L_{p_1}$ and $\|\mathfrak{h}u-\mathfrak{h}v\|_B\to 0$ if $\|u-v\|_B\to 0$. Theorem 1. In order that $\mathfrak{h}u_m\to\mathfrak{h}u_0$ in L_{p_1} follow from $u_m\to u_0$ in L_p, it is necessary and sufficient that for each $\epsilon>0$ there is an $\eta>0$ such that for each measurable $F\subset B$ with meas $F<\eta$ it follows that $\|\mathfrak{h}u\|_F<\epsilon$. Theorem 2. In order that $\mathfrak{h}$ be continuous from L_p into L_{p_1} it suffices that it be compact, that is, that it take bounded sets in L_p into compact sets in L_{p_1}. Theorem 3. In order that $\mathfrak{h}$ be continuous from L_p into L_{p_1} it suffices that $f(u,x)\leqq a(x)+b|u|^{p/p_1}$, where $a(x)\in L_{p_1}$ and b is a constant.

After proving these results, the author generalizes u to represent a vector $u_1,\cdots,u_m$, and states the following theorem on integral equations. Assume (1) the functions $f_i(u_1,\cdots,u_m,x)$ continuous in u and measurable in x, and $|f_i(u_1,\cdots,u_m,x)|\leqq a_i(x)+\sum_{r=1}^m b_{ir}|u_r|^{p-1}$, $i=1,2,\cdots,m$, where $a_i(x)\in L_q$, $1/p+1/q=1$, b_{ir} constant;

$$(2)\qquad \int_B\int_B|K_i(x,y)|^p dxdy=M_i^p<\infty,$$

$i=1,2,\cdots,m$, where $K_i(x,y)$ is a real function measurable in the topological product $B\times B$; (3) either $p<2$, or $p=2$ and $\sum_{i=1}^m\sum_{r=1}^m M_i^2 b_{ir}^2<1$, or $p>2$ and

$$\left(\sum_{i=1}^m M_i^p\|a_i\|^p\right)^{1/p}+\left(\sum_{i=1}^m\sum_{r=1}^m M_i^p b_{ir}^p\right)^{1/p}\leqq 1,$$

where $\|a_i\|=(\int_B|a_i(x)|^q dx)^{1/q}$; then the system of nonlinear integral equations $u_i(x)=\int_B K_i(x,y)f_i(u_1(y),\cdots,u_m(y),y)dy$, $i=1,2,\cdots,m$, has at least one solution belonging to the space $L_{p,m}$. *M. M. Day* (Urbana, Ill.).

Berman, D. L. On some linear operations. Doklady Akad. Nauk SSSR (N.S.) **73**, 249–252 (1950). (Russian)

Let C denote the space of continuous real-valued functions defined on the closed interval $[-1,+1]$. Let $U_n^{[k]}(f,x)$ be a linear operator carrying C into C in such a manner that (1) $U_n^{[k]}(f,x)$ is a polynomial of degree $\leqq n$, (2) $U_n^{[k]}(P,x)$ is the kth derivative of P if P is a polynomial of degree $\leqq n$. As an example of such operators, the author submits the following. Let $\{\omega_n(x)\}_{n=0}^\infty$ be an orthonormal system of polynomials on $[-1,+1]$ with weight function $g(x)\geqq 0$. Then the kth derivative of the nth partial sum of the Fourier series of $f\in C$ (with respect to the family $\{\omega_n(x)\}_{n=0}^\infty$) is an operator of the required type. A second example, which is not clearly explained, is also produced. Two theorems concerning the operator $U_n^{[k]}(f,x)$ are stated, and proofs for both are sketched. The first proof is based on an incorrect formula for the derivative $(d^k/dx^k)\cos(n(\arccos x))$. The second proof, owing to obscurities in the notation employed, is not clear to the reviewer. *E. Hewitt.*

Naĭmark, M. A. Rings with involution. Amer. Math. Soc. Translation no. **25**, 131 pp. (1950).

Translated from Uspehi Matem. Nauk (N.S.) **3**, no. 5(27), 52–145 (1948); these Rev. **10**, 308. The missing equation on p. 8 is $A\xi^{(\alpha)}=\{A_{\beta\alpha_s}\xi_{\alpha_s}\}$.

Šilov, G. E. On a theorem of I. M. Gel'fand and its generalizations. Doklady Akad. Nauk SSSR (N.S.) **72**, 641–644 (1950). (Russian)

Let y be a generalized nilpotent element in a normed ring R with identity element e and let $x=e-y$, $\alpha_n=\|x^n\|$. Gelfand [Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 49–50 (1941); these Rev. **3**, 36] proved that, if $\alpha_n=O(1)$ for $n=0,\pm 1,\pm 2,\cdots$, then $y=0$. Hille [Proc. Nat. Acad. Sci. U. S. A. **30**, 58–60 (1944); these Rev. **5**, 189] improved this result by replacing $O(1)$ by $o(1)$. Stone [J. Indian Math. Soc. (N.S.) **12**, 1–7 (1948); these Rev. **10**, 308] extended the result further by showing that a necessary and sufficient condition for $y^{N+1}=0$ (resp. $y^N=0$) is that $\alpha_n=O(|n|^N)$ (resp. $\alpha_n=o(|n|^N)$). In the present note the author observes that the above results are contained in the following theorem of Gelfand [Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 41–48 (1941); these Rev. **3**, 52]. Let R_0 be a normed ring generated by x and x^{-1}, and for $\alpha_n=\|x^n\|$ assume

$$(1)\qquad \lim_{r\to 1}(1-r)^k\sum_{n=0}^{\infty}\alpha_n r^n=0,\quad \lim_{r\to 1}(1-r)^k\sum_{n=0}^{\infty}\alpha_{-n}r^n=0.$$

Then each maximal ideal $M_0\subset R_0$ contains at most $k-1$

distinct primary ideals $I_1, \cdots, I_{k-1}$, where I_j is generated by $(x-x(M_0)e)^j$. Moreover, the element $(x-x(M_0)e)^{k-1}$ is contained in every primary ideal of R_0. [This last statement is only implicit in Gelfand's paper.] As an illustration consider the case discussed by Stone and let R_0 be the ring generated by $(e-y)^{\pm 1}$. Then $\alpha_n = O(|n|^N)$ implies condition (1) for $k=N+2$. Furthermore, R_0 has a unique maximal ideal M_0, $x(M_0)=1$, and the zero ideal is primary in R_0. It follows from the last statement of the Gelfand theorem that $y^{N+1}=0$. The case $\alpha_n = o(|n|^N)$ is handled similarly. The paper also contains an example to show that the condition $\alpha_n = O(1)$ for positive n is not sufficient for the first Gelfand result. *C. E. Rickart* (New Haven, Conn.).

Eberlein, W. F. Abstract ergodic theorems and weak almost periodic functions. Trans. Amer. Math. Soc. 67, 217–240 (1949).

This paper consists of two parts. The first part is on general abstract ergodic theorems concerning semi-groups $G=\{T\}$ of bounded linear operators on a Banach space, and gives the details of the proofs of the results previously announced by the author [Proc. Nat. Acad. Sci. U. S. A. 34, 43–47 (1948); these Rev. 9, 359]. One of the main points of the author's ergodic theorem is that the boundedness of $\|T\|$ for $T \in G$ is not required. This makes it possible to apply it to Fejér's summation in Fourier series. In fact, the operators U_n defined on the space of real-valued continuous functions $x(t)$ of a real variable of period 2π with the norm $\|x\| = \sup_{0 \leq t \leq 2\pi} |x(t)|$ by $U_n x(t) = s_n(t)$ (i.e., the nth partial sum of the Fourier series of $x(t)$) form an Abelian semi-group $(U_m U_n = U_{\min(m,n)})$ with $\|U_n\| = (4/\pi^2) \log n + O(1)$, while the arithmetic means $T_n = (1/n)\sum_{k=0}^{n-1} U_k$ satisfy $\|T_n\| \leqq 2$, $n=1, 2, \cdots$. Uniform ergodic theorems concerning semi-groups of compact or quasi-compact operators are also discussed.

In the second part, the author considers weakly almost periodic (a.p.) functions on a locally compact Abelian group G. Let $C(G)$ be the Banach space of all bounded continuous complex-valued functions $x(t)$ defined on G with the norm $\|x\| = \sup_{t \in G} |x(t)|$. A function $x(t) \in C(G)$ is weakly a.p. if the set of all translations $\{x(t+s) \mid s \in G\}$ is conditionally weakly compact in $C(G)$. The class W of all weakly a.p. functions is considerably wider than the class of all strongly a.p. functions (i.e., ordinary a.p. functions), and W includes all positive definite functions and all functions which vanish at infinity. Every function from W is uniformly continuous and the existence of the mean value (in the strong topology) for weakly a.p. functions follows immediately from the ergodic theorem proved in the first part. The author further shows that the convolution of two weakly a.p. functions is strongly a.p., that the Parseval relation holds for weakly a.p. functions, and finally, in case G is the additive group of real numbers with the ordinary topology, that every function from W is Weyl's a.p. function of index p for any $p \geqq 1$. The determination of the class W in terms of Fourier analysis on G is left open. *S. Kakutani.*

Calculus of Variations

Faedo, S. Una osservazione sul metodo di Ritz. Ann. Scuola Norm. Super. Pisa (3) 2 (1948), 85–97 (1950).

Soit $y(x)$ une fonction continue dans (a, b) avec sa dérivée $y'(x)$ et Γ un ensemble de fonctions $y(x)$. Si, avec la métrique définie par la formule

$$(y_1, y_2) = \max_{a \leqq x \leqq b} \{|y_1(x)-y_2(x)| + |y_1'(x)-y_2'(x)|\},$$

l'ensemble Γ est fermé et compact et si $f(x, y, y')$ est une fonction continue dans le domaine $a \leqq x \leqq b$, $-\infty < y, y' < +\infty$, on sait que l'intégrale $I(y) = \int_a^b f(x, y, y')dx$ admet minimum (et maximum) absolu dans Γ. L'auteur montre que, dans ce cas, la construction de suites minimisantes $I(y)$ dans Γ peut s'effectuer avec la méthode de Ritz (quand même f n'est pas quadratique en y, y'), pourvu que les $y(x)$ de Γ aient la deuxième dérivée $y''(x)$ de carré sommable dans (a, b). Cette thèse est prouvée sur plusieurs exemples de tels ensembles Γ; nous expliquons la méthode dans un des cas envisagés. Soit Γ_1 l'ensemble des fonctions $y(x)$ absolument continues dans (a, b) et satisfaisantes aux conditions $y(a)=c_1$, $y'(a)=c_2$, $\int_a^b y'^2(x)dx \leqq L$. Si $\{\psi_n(x)\}$, $n=1, 2, \cdots$, est un système orthonormal et complet dans (a, b), on doit poser

$$y_m(x) = y_m(x; \lambda_1, \cdots, \lambda_m) = c_1 + c_2(x-a) + \sum_{n=1}^{m} \lambda_n \int_a^x (x-\xi)\psi_n(\xi)d\xi;$$

$$I(\lambda_1, \cdots, \lambda_m) = I(y_m) = \int_a^b f[x, y_m(x), y_m'(x)]dx;$$

alors, si $(\lambda_1^*, \cdots, \lambda_m^*)$ est un point de minimum absolu de $I(\lambda_1, \lambda_2, \cdots, \lambda_m)$ dans l'hypersphère $\sum_{n=1}^{m} \lambda_n^2 \leqq L$, la fonction $y_m^*(x) = y_m(x; \lambda_1^*, \cdots, \lambda_m^*)$ décrit, pour $m=1, 2, \cdots$, une suite minimisante $I(y)$ dans Γ_1. *A. Ghizzetti* (Rome).

Balada, Emilio. Il problema isoperimetrico del calcolo delle variazioni. Ann. Scuola Norm. Super. Pisa (2) 15 (1946), 97–112 (1950).

Let $\mathfrak{F}(\mathfrak{C}) = \int_{\mathfrak{C}} F(x, y, x', y')ds$ and $\mathfrak{G}(\mathfrak{C}) = \int_{\mathfrak{C}} G(x, y, x', y')ds$ be two integrals of the calculus of variations for plane curves in parametric form; the problem is to find in a given class K of curves one which minimizes $\mathfrak{F}(\mathfrak{C})$ subject to the condition that $\mathfrak{G}(\mathfrak{C})$ have an assigned value L. The reviewer obtained some theorems for this problem [Trans. Amer. Math. Soc. 44, 429–438, 439–453 (1938); ibid. 45, 151–171, 173–216 (1939)]. The present paper obtains (for problems in the plane) more general results by very different and much less intricate methods; the results are more closely related to those obtained by the reviewer (as special cases of Bolza problems) with the help of the theory of generalized curves [Duke Math. J. 7, 28–61 (1940); these Rev. 2, 226], but are not identical. The hypotheses are too long to state in full, so we state only the most distinctive assumption; for each λ let H be the combination $F+\lambda G$; at each (x, y), each plane tangent to the graph of H touches it at elements in a section bounded by rays whose angles we call ϑ_1, ϑ_2; the assumption is that

$$H_x(x, y, \cos\vartheta_1, \sin\vartheta_1) + H_y(x, y, \cos\vartheta_1, \sin\vartheta_1) - H_x(x, y, \cos\vartheta_2, \sin\vartheta_2) - H_y(x, y, \cos\vartheta_2, \sin\vartheta_2) \neq 0.$$

An important auxiliary in the proof is a functional

$$\mathfrak{R}(\mathfrak{C}) = \mathfrak{F}(\mathfrak{C}) + \lambda\mathfrak{G}(\mathfrak{C}) + \Phi(\mathfrak{G}(\mathfrak{C})),$$

λ real, Φ a continuous differentiable and increasing function on the real numbers. This is first shown to have a minimizing curve. Corresponding to given λ and Φ, the minimizing curves give values to $\mathfrak{G}(\mathfrak{C})$; as λ, Φ vary, the corresponding $\mathfrak{G}(\mathfrak{C})$ form a set of numbers dense in an interval containing L. A continuity proof shows that there exists a

λ_1 and Φ_1 such that $\mathfrak{F}(\mathfrak{C})+\lambda_1\mathfrak{G}(\mathfrak{C})+\Phi_1(\mathfrak{G}(\mathfrak{C}))$ assumes its minimum on a curve $\mathfrak{C}_0$ such that $\mathfrak{G}(\mathfrak{C}_0)=L$; $\mathfrak{C}_0$ then minimizes $\mathfrak{F}$ on the class of curves for which $\mathfrak{G}(\mathfrak{C})=L$.

E. J. McShane (Charlottesville, Va.).

Cinquini, Silvio. Sopra l'estremo assoluto degli integrali doppi in forma ordinaria. Ann. Mat. Pura Appl. (4) **30**, 249–260 (1949).

The author gives two theorems on the existence of a minimum for integrals of the form

$$\int\int_D F(x, y, z(x, y), \partial z/\partial x, \partial z/\partial y)dx\,dy,$$

where the hypothesis $F\geqq$constant is replaced by a condition of the form $F(x, y, z, p, q)\geqq M_1\{|p|^\mu+|q|^\mu\}-M_2|z|^\nu+N$, where $M_1>0$, $0<\nu<\mu>2$. In the first theorem admissible functions $z(x, y)$ are supposed bounded and equicontinuous on the boundary of the domain D of integration; in the second theorem most of this restriction is replaced by a restriction on the shape of D. The proofs follow the methods of Tonelli [Ann. Scuola Norm. Super. Pisa (2) **2**, 89–130 (1933)] with use of some supplementary inequalities. A few counterexamples are included. *L. M. Graves.*

Stampacchia, Guido. Gli integrali doppi del calcolo delle variazioni, in forma ordinaria. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 21–26 (1950).

Let Σ be a space of functions u defined on an open set T in the plane, Σ being metrized by a distance function (u_1, u_2). Functions $F_i(u, R)$, $i=1, \cdots, n$, are defined for all u in Σ and all rectangles R contained in T, continuous on Σ for fixed R; $\mathfrak{G}$ is the subclass of Σ for which the $F_i(u, R)$ are absolutely continuous functions of rectangles in T, in which case their derivatives are called $\mathfrak{D}_iu(x, y)$. By proper choice of F_i, $\mathfrak{G}$ becomes the class of functions absolutely continuous in the sense of Tonelli, and the $\mathfrak{D}_i$ their partial derivatives; other special cases are also considered. For $\alpha>0$, let φ_α be a positive continuous function on n-space such that, when $\mathfrak{J}$ is a subset of $\mathfrak{G}$ on which

$$N_\alpha[u]=\int\int_T \varphi_\alpha(\mathfrak{D}_1u(x, y), \cdots, \mathfrak{D}_nu(x, y))dx\,dy$$

is bounded, so are the integrals of the $|\mathfrak{D}_iu(x, y)|^{1+\alpha}$. If $\mathfrak{J}\subset\mathfrak{G}$ and N_α is bounded on $\mathfrak{J}$, the closure of $\mathfrak{J}$ in Σ is contained in $\mathfrak{G}$. If $f(x, y, u, w_1, \cdots, w_n)$ is continuous with its partials as to the w_i for (x, y) in T and all u, w, and is convex in w and bounded below, its integral is lower semi-continuous on every class $\mathfrak{J}\subset\mathfrak{G}$ on which N_α is bounded. If $f(x, y, u, w_1, \cdots, w_n)>\mu\varphi_\alpha(w_1, \cdots, w_n)+N$ for some positive μ and α, and $\mathfrak{J}$ is a subclass of $\mathfrak{G}$ on which f is integrable, and every subset of $\mathfrak{J}$ on which N_α is bounded is compact, then the integral of f assumes its minimum.

E. J. McShane (Charlottesville, Va.).

***Tonelli, Leonida. L'analisi funzionale nel calcolo delle variazioni.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 373–387, Rome, 1943.

This paper gives a brief outline in very general terms of the development of the calculus of variations and of its relationship with the functional calculus. The closing section lists problems most of which have been studied with some success since this paper was written. Among these are the problems of Mayer and Bolza, the minimum of double integrals depending on a function of one variable, and the minimum of functionals of parametric surfaces.

L. M. Graves (Chicago, Ill.).

Theory of Probability

Tintner, Gerhard. Foundations of probability and statistical inference. J. Roy. Statist. Soc. Ser. A. **112**, 251–279; discussion, 280–286 (1949).

An exposition of R. Carnap's theory of probability [Philos. Sci. **12**, 72–97 (1945); Philos. and Phenomenol. Res. **5**, 513–532 (1945); these Rev. **7**, 46, 189]. Only the case of finite language systems and its applications to statistical inference is discussed. The following remark of the author is of interest in connection with the following review. Assuming that the probability p of a success in a single trial has a rectangular distribution in the interval (0, 1), we obtain $\int_0^1\binom{n}{s}p^s(1-p)^{n-s}dp=1/(n+1)$. This is Carnap's measure for a structure description in a language with n individuals and one attribute only [see the following review].

J. Łoś (Wrocław).

James, S. F. A note on Carnap's theory of probability. J. Roy. Statist. Soc. Ser. A. **112**, 309–315 (1949).

In connection with a remark of G. Tintner [see the preceding review] the author demonstrates that in the case of two independent predicates and two individuals, the probability of a structure description given by the integral

$$\int_0^1 dp_1\int_0^1 dp_2 m p_1^{s_1}(1-p_1)^{n-s_1}\cdot p_2^{s_2}(1-p_2)^{n-s_2}$$

(where n is the number of individuals, in this case 2, m is the number of state descriptions in the structure description, and p_1, p_2 are the probabilities of the first and second predicates respectively) differs from that of Carnap in this case [Philos. Sci. **12**, 72–97 (1945); these Rev. **7**, 46]. Another theory of probability is proposed which is similar to Carnap's, but has a closer link with the classical theory.

J. Łoś (Wrocław).

Bose, R. C. On a problem of two dimensional probability. Sankhyā **10**, 13–28 (1950).

A rectangle of side mn is divided into mn squares. The squares are colored black and white at random with probabilities q and $p=1-q$ respectively. Two white squares are called connected if they occur in a chain of white squares having contact along a side. Two black squares are called connected if they occur in a chain of black squares having contact along a side or at one corner. Squares which are not connected are called separated. A set of squares connected with each other and separated from the remaining squares of like color is called a patch. Let X be the number of white patches decreased by the number of black patches which are entirely bordered by white patches (that is to say, have no contact with the border of the rectangle). The author's aim is to find $E(X)$ and $E(X^2)$. Denote a square by (i, j) if its upper right hand corner has coordinates (i, j), the coordinate axes running along the lower and left hand sides of the rectangle. In the following definitions consider the outside of the rectangle as black. If (i, j) is white, then (i, j) is called a corner if $(i-1, j)$, $(i, j-1)$ is black; a junction if $(i-1, j)(i, j-1)$ are white but $(i-1, j-1)$

black. Let N_s be the number of corners, N_j the number of junctions. The author shows $X=N_s-N_j$ and then obtains $E(X)$ and Var (X) by computing the expectations, variances, and the covariance of N_s and N_j. *H. B. Mann.*

Domb, C. Some probability distributions connected with recording apparatus. II. Proc. Cambridge Philos. Soc. **46**, 429–435 (1950).

An apparatus registers two types of events occurring independently according to two Poisson distributions. Each recording is followed by a dead period of fixed length τ. The author says that a coincidence occurs whenever an event occurs during the dead period following a registration of an event of the opposite type. [This should be distinguished from coincidences in the usual sense which refer to two apparatuses and require more delicate mathematical techniques.] Let $x_{n1}(t)$ be the probability that during time t there occur n coincidences and the apparatus is free at time t. It is easy to derive a difference-differential equation for $x_{n1}(t)$ which can be solved by Laplace transforms. From $x_{n1}(t)$ it is possible to calculate $x_n(t)$, the probability of n coincidences.

W. Feller (Princeton, N. J.).

Kac, Mark, and Pollard, Harry. The distribution of the maximum of partial sums of independent random variables. Canadian J. Math. **2**, 375–384 (1950).

Let $p(a, t)=\lim_{n\to\infty} P\{\max_{k<n} |S_k|<an\}$, where S_k are partial sums of a series of independent and identically Cauchy distributed random variables. The authors prove that $p(a, t)$ exists and express it as a series whose terms depend upon eigenvalues and eigenfunctions of an integral equation of Hilbert-Schmidt type; the mean time of absorption for the corresponding Cauchy process is computed.

M. Loève (Berkeley, Calif.).

Brunk, H. D. Note on a theorem of Kakutani. Proc. Amer. Math. Soc. **1**, 409–414 (1950).

The author applies a theorem of Kakutani [Ann. of Math. (2) **49**, 214–224 (1948); these Rev. **9**, 340] concerning the equivalence of two direct infinite measures and obtains a device which contains that of Markoff-Khintchine truncation in the case of series of independent random variables.

M. Loève (Berkeley, Calif.).

Loève, M. Fundamental limit theorems of probability theory. Ann. Math. Statistics **21**, 321–338 (1950).

This paper overlaps to some extent one by Feller [Bull. Amer. Math. Soc. **51**, 800–832 (1945); these Rev. **7**, 128]. However, while Feller amplified on the central limit theorem and the law of the iterated logarithm, here the stress is laid on more general limit theorems, specifically the theory of the infinitely divisible laws and the case of dependent variables. In the former category not only the classical Lévy formula and the characterizations of Lévy and Khintchine, but also the necessary and sufficient conditions due to Gnedenko and Doblin, which include those for the central limit theorem as a special case, are discussed. In the second category the results of S. Bernstein and Lévy, and the author's own contributions are discussed. Historical and methodological remarks help clarify the issues. The title has restricted the scope of this paper so that little is mentioned of various related developments since Feller's address five years ago.

K. L. Chung (New York, N. Y.).

Montroll, Elliott W. Markoff chains and excluded volume effect in polymer chains. J. Chem. Phys. **18**, 734–743 (1950).

The author studies a random walk on a square plane lattice, in which the moving particle can only move (in single steps) to the left or right, and cannot make three successive left or right moves. The random walk is treated as a multiple Markov chain in which the conditional probability distribution of a step in terms of past steps depends on the two, but no more, preceding steps. An asymptotic expression (of order n) is found for the mean square distance between the starting point and the point reached in n steps. The application to polymer theory is discussed.

J. L. Doob (Urbana, Ill.).

***Lévy, Paul. Processus doubles de Markoff. Le Calcul des Probabilités et ses Applications.** Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 53–59. Centre National de la Recherche Scientifique, Paris, 1949.

Let $X(M)$ be a real-valued random variable depending continuously (in the sense of convergence in probability) on a parameter M which runs through all points of the plane. Then $X(M)$ is called degenerate if, for any M_0 and for any circular neighborhood $V(M_0)$ of M_0, the values of $X(M)$ at $M=M_0$ are uniquely determined by those of $X(M)$ for M outside of $V(M_0)$. Also, $X(M)$ is called a double Markoff process, if for any line L which separates the plane into two regions R_1, R_2, the values of $X(M)$ for $M\varepsilon R_1$ and those of $X(M)$ for $M\varepsilon R_2$ are independent under the condition that we know the values of $X(M)$ for M on L. More precisely, if we denote by Pr $(A\,|\,L)$ the conditional probability of the event A when the values of $X(M)$ for M on L are known, then Pr $(A\cap B\,|\,L)=$ Pr $(A\,|\,L)$ Pr $(B\,|\,L)$ for any A, B of the form:

$$A=\{X(M_i)<x_i\,|\,i=1, \cdots, p\},$$
$$B=\{X(N_j)<y_j\,|\,j=1, \cdots, q\},$$

where $M_i\varepsilon R_1$, $N_j\varepsilon R_2$, and x_i, y_j are real numbers, $i=1, \cdots, p$; $j=1, \cdots, q$. The examples announced previously by the author [C. R. Acad. Sci. Paris **220**, 420–422 (1945); **226**, 53–55, 307–308 (1948); these Rev. **7**, 130; **9**, 361] led the author to the conjecture that all double Markoff processes are degenerate. In this paper the author shows that there exists a nondegenerate double Markoff process. It seems, however, as the author observes, that the author's conjecture remains true if we generalize the notion of degeneracy in a suitable way. *S. Kakutani* (New Haven, Conn.).

Mourier, Édith. Propriétés des caractéristiques d'un élément aléatoire dans un espace de Banach. C. R. Acad. Sci. Paris **231**, 28–29 (1950).

This note continues the discussion [C. R. Acad. Sci. Paris **229**, 1300–1301 (1949); these Rev. **11**, 376] of the characteristic function of a probability distribution defined on sets of a Banach space, that is of the characteristic function of a certain type of abstract-valued random variable. The characteristic function ϕ is a function defined on the conjugate space. Various properties of ϕ are stated without proof. For example: ϕ is continuous in the weak topology of the conjugate space: ϕ is positive definite, and essentially determines the given probability measure. If the given Banach space is separable and reflexive, a necessary and sufficient condition is stated that a positive definite function ϕ defined on the conjugate space and vanishing at the origin be a characteristic function. *J. L. Doob* (Urbana, Ill.).

Damle, S. C. The $\bar{Z}$ method for the expression of multiple-life contingencies in terms of last survivor statuses. J. Inst. Actuaries Students' Soc. **9**, 286–295 (1950).

The so called Z-formulae in actuarial mathematics give concise expressions for certain life contingencies. These are represented by means of a linear basis. In the Z-method the basis is the system of the joint life probabilities ${}_t p_x$, ${}_t p_{xy}$, ${}_t p_{xyz}$, $\cdots$. The author modifies the Z-method by using as the basis instead of the joint survivor probabilities the last survivor probabilities ${}_t p_x$, ${}_t p_{\overline{xy}}$, ${}_t p_{\overline{xyz}}$, $\cdots$ and obtains the completely analogous $\bar{Z}$-formulae. Both the Z and $\bar{Z}$-method apply also to annuities and assurances. *E. Lukacs.*

Rybarz, Josef. Zum Hattendorffschen Satz. Statist. Vierteljschr. **2**, 32–36 (1949).

Hattendorff's theorem can be stated and proved in two different ways, according as the discrete or the continuous method is used. [For a survey see A. Berger, Assekuranzjahrbuch **50**, 18–39 (1931).] The author employs Stieltjes integrals to give a unified formulation and proof for both variants of Hattendorff's theorem.

E. Lukacs (Washington, D. C.).

Mathematical Statistics

David, F. N., and Johnson, N. L. The probability integral transformation when the variable is discontinuous. Biometrika **37**, 42–49 (1950).

H. O. Lancaster [Biometrika **36**, 370–382 (1949)] investigated arithmetically the behavior of the cumulative frequency y for a discrete variable x as a random variable. The present authors undertake a theoretical investigation in the case the population parameters in the discontinuous law for x are known, as was assumed by Lancaster, and also in the case that these population parameters are estimated from a sample. In the first case the first four moments of y are found and applications are made in the cases in which x obeys a binomial and a Poisson discontinuous law. The effect on R. A. Fisher's application of χ^2 to the probability integral transformation is also studied. In the second case it is assumed that only the mean is fitted to the data and a general expression for the moments of y is found. The special forms assumed when x obeys a binomial and a Poisson distribution law are given. *C. C. Craig.*

Hartley, H. O. A simplified form of Sheppard's correction formulae. Biometrika **37**, 145–148 (1950).

For a random variable x obeying a probability distribution function $f(x)$ with continuous derivatives up to order $2k-1$ for $a \leqq x \leqq b$, the author uses the Euler-Maclarin formula to express the rth grouped moment about 0 in terms of the $(r-1)$st moment of a variable whose discrete set of values are the left end points of class intervals, with corresponding frequencies the complements of the cumulative frequencies at these end points, and in terms of the derivatives of these frequencies at a and b plus a remainder term. If enough derivatives vanish at a and b, and if the remainder term is negligible, the results of course agree with the usual Sheppard formulae. If there is not high contact a scheme is developed for calculating the terms involving the derivatives and for majorizing the error in terms of the highest derivative used. For a and b infinite it is suggested that a truncation procedure be used with the contribution to the grouped moment beyond the truncation points negligible. Essentially the computational form is what is new here; as is usual in papers on this subject the behavior of the remainder term in specific cases is not carefully examined.

C. C. Craig (Ann Arbor, Mich.).

Lyttkens, Ejnar. On a class of conditional distributions, connected with some astronomical problems. Ark. Astronomi **1**, 11–25 (1950).

Lyttkens, Ejnar. On a class of multidimensional conditional characteristic functions and semi-invariants. Ark. Astronomi **1**, 27–45 (1950).

Lyttkens, Ejnar. A generalisation of the multidimensional A-series. Ark. Astronomi **1**, 47–57 (1950).

Lyttkens, Ejnar. Determination of unknown distributions by means of conditional frequency functions. Ark. Astronomi **1**, 69–75 (1950).

Let $X=Y+Z$ where Y and Z are independently distributed statistical variables. The problem which is the principal concern of all four of this series of papers is to find the conditional distribution of Y for a given value of X, or for X lying in a specified interval, when only the frequency functions (fr. f.'s) X and Z are known. The particular application of interest to the author is that in which X is the apparent magnitude of a star, Z the absolute magnitude and Y is the difference between the apparent and the absolute magnitude. The closely connected fundamental equation of stellar statistics connecting the distributions of apparent magnitudes, distances, and absolute magnitudes, can be put in the form of the convolution which is the author's point of departure.

The greater part of the first paper is devoted to a rederivation of K. G. Malmquist's results [Medd. Lunds Astron. Obs. Ser. II, no. **22** (1920); Ark. Mat. Astr. Phys. **16**, no. 23 (1922); **25A**, no. 14 (1936)] for the conditional semi-invariants (c.s.i.'s) of Y for a given X, for $X \leqq x$ and for X in a given interval, it being assumed that the fr. f. of z is normal. These results give the coefficients in the type A series in which it is assumed the conditional frequency of Y may be expressed. The author makes use of the corresponding conditional characteristic functions (c.c.f.'s). The more general point c.c.f. and c.s.i.'s of X, Y, and Z for $X=x$ and $X \leqq x$ are also found. A final section gives derivations of the c.c.f. and the c.s.i.'s of Y for $X=x$ when the fr. f. of z is expressed as the first three terms of a type A series.

In the second paper the methods and results in the first are extended to the case when X, Y, and Z are s-dimensional variables, the given distribution of $Z=(Z_1, \cdots, Z_s)$ being s-variable normal. The c.c.f.'s and c.s.i.'s for both Z and Y for $X=x$ and X in a given region are found. The special cases in which the components of Z are uncorrelated and in which the variables are two-dimensional are considered.

In the third paper the author formally generalizes the multivariate type A series by replacing the normal multivariate generating function by an arbitrary multivariate fr. f. Assuming the validity of such an expansion, he finds the c.f., and expressions for the coefficients in terms of moments and semi-invariants. If now x is the vector sum of two independent s-dimensional variables Y and Z (e.g., x is the observed value of X, Y the true value, and Z the error of observation), the author makes a formal application by expressing the fr. f. of y, using that for x as the generating function for a given fr. f. of z.

The fourth paper returns to the problem of the second in which x, y, and z are s-dimensional and in which now the conditional fr. f. of $x-y$ for given x is employed. It is assumed that the fr. f. of z is normal and then that the logarithm of the fr. f. of x may be approximated in certain regions (1) by a polynomial of the 2d degree, (2) by a polynomial of the 1st degree. It turns out that for the fr. f. in such regions, the conditional fr. f. of y can be expressed in like form. *C. C. Craig* (Ann Arbor, Mich.).

Kolmogorov, A. N. Unbiased estimates. Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 303–326 (1950). (Russian)

The purpose of this expository article is to stimulate further work on what might be called abstract statistics, and specifically on the theories of unbiased and sufficient estimation. Writing, apparently, for a reader with more training in practical statistics than in pure mathematics, the author defines unbiased estimates and sufficient statistics, and obtains some of their elementary properties. He discusses in great (and even numerical) detail some examples involving binomial and normal distributions (the former in the language of the theory of quality control). Although, according to the author, the paper presumes to do no more than to demonstrate by examples that this part of statistics deserves more attention than it has received, this modesty is not fully justified. In the introduction, for instance, Blackwell's results on improving an unbiased estimate by means of a sufficient statistic [Ann. Math. Statistics **18**, 105–110 (1947); these Rev. **8**, 478] are derived in a slightly generalized form, and these generalizations are later applied to simplify the proofs of some of the reviewer's results on symmetric unbiased estimates [Ann. Math. Statistics **17**, 34–43 (1946); these Rev. **7**, 463]. *P. R. Halmos.*

Katz, Leo. On the relative efficiencies of BAN estimates. Ann. Math. Statistics **21**, 398–405 (1950).

Neyman [Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 239–273, University of California Press, Berkeley-Los Angeles, 1949, p. 239; these Rev. **10**, 388] has defined a class of BAN estimates which include maximum likelihood estimates and modified χ^2-minimum estimates. The application of the latter usually involves truncation of the probability law. The effect of this truncation is studied here for the class of discrete probability distributions (the points with positive probability may, without loss of generality, be taken to be the positive integer points) which in addition possess an "efficient" statistic in the sense of Cramér [Mathematical Methods of Statistics, Princeton University Press, Princeton, N. J., 1946, p. 481; these Rev. **8**, 39]. For this class of distributions it is shown that (1) the variance of the maximum likelihood estimate is smaller than the variance of the modified χ^2-minimum estimate; (2) the variance of the modified χ^2-minimum estimate is a decreasing function of the truncation point. Some examples are discussed and these results are generalized to multiparameter distributions satisfying corresponding conditions. *D. G. Chapman.*

Davis, R. C. Derivation of a broad class of consistent estimates. Ann. Math. Statistics **21**, 425–431 (1950).

The author gives sufficient conditions under which he can construct consistent estimates of a vector-valued parameter θ. A typical result states as the main assumption the existence of a function g of the random variable X and of θ, whose expectation is 0 and which for some $0<K<1$ satisfies $\|g(X,\theta_1)-g(X,\theta_2)-(\theta_1-\theta_2)\| \leqq K\|\theta_1-\theta_2\|$. No examples are given. *E. L. Lehmann* (New York, N. Y.).

Nair, K. R. Efficiencies of certain linear systematic statistics for estimating dispersion from normal samples. Biometrika **37**, 182–183 (1950).

If $x_1, x_2, \cdots, x_n$ is a random sample of n observations from a normal population arranged in ascending order of magnitude, let

$$j_{(r)}=(x_n+x_{n-1}+\cdots+x_{n-r+1})-(x_r+x_{r-1}+\cdots+x_1).$$

The efficiency of $j_{(r)}$ as an estimate of the standard deviation is obtained and compared with the most efficient linear systematic statistic for $n=2$ to 10 (the values of n for which the efficiency of this latter statistic is available). *H. Chernoff* (Urbana, Ill.).

Pearson, E. S. Some notes on the use of range. Biometrika **37**, 88–92 (1950).

The author investigates the estimation of the standard deviation of a population by use of the range of a sample or the mean range of a sample of N items subdivided into m groups of n items each (a) if the population is not normal, or (b) if the sample includes one or more outlying observations which may upset the range as an estimator of the standard deviation. Basing his conclusions on previous experimental and theoretical results, he feels that the case for using range, with adjustment appropriate for a normal population, as an estimator of standard deviation is well supported provided that n is chosen as ten or less. *L. A. Aroian* (Culver City, Calif.).

Patnaik, P. B. The use of mean range as an estimator of variance in statistical tests. Biometrika **37**, 78–87 (1950).

Let $x_1, x_2, \cdots, x_n$ be a random sample of n observations, arranged in ascending order of magnitude, drawn from a normal population with mean μ and standard deviation σ. Define the range by $w_n=x_n-x_1$ and, if there are m such independent samples with n observations each, define the mean of the m ranges by $\bar{w}_{m,n}$. Further d_n is defined by $E(w_n)=\sigma d_n$. The first two moments of $\bar{w}_{m,n}$ are used to approximate the distribution by means of a χ distribution; specifically, $\bar{w}_{m,n}$ is distributed approximately as cs, where s is the sample root-mean-square estimate of σ based on ν degrees of freedom, and c is given by

$$d_n\left\{1+\frac{1}{4\nu}+\frac{1}{32\nu^2}+\frac{5}{128\nu^3}\right\}.$$

Lord [see the following review] has derived the distribution of $u=xd_n/\bar{w}_{m,n}$ numerically, where x is distributed normally with mean zero and standard deviation σ. The author shows that u is approximately distributed as d_n/c times Student's t. Numerical checks of this approximation against Lord's results for various percentage levels show it to be excellent, and the same is true for the power function of u under various alternatives. The author also obtains an analogue of the F test in the analysis of variance and discusses its power for suitable alternatives. Cox [J. Roy. Statist. Soc. Ser. B. **11**, 101–114 (1949); these Rev. **11**, 262] has approximated the distribution of the range by means of a χ^2 distribution with the correct first two moments. Cox is mainly concerned with applications in sequential analysis. *L. A. Aroian* (Culver City, Calif.).

Lord, E. Power of the modified t-test (u-test) based on range. Biometrika **37**, 64–77 (1950).

Previously the author [Biometrika **34**, 41–67 (1947); these Rev. **8**, 394] discussed the use of the u test, $u=$(normal deviate)/(independent range or mean-range estimate of standard error), for a normal population with mean μ and variance σ^2 in testing various hypotheses concerning a population mean or finding confidence limits for the mean. In this paper the power of the u-test is determined and this power is compared with the power of the uniformly most powerful unbiased test of Student's hypothesis. Two cases arise: (a) if the standard deviation is estimated by the sample range of N items; or (b) if the standard deviation is estimated by subdividing the N items into m groups of n each and then using the mean range of the m groups. In both cases the u test is only slightly less powerful than Student's "t" test. The results are particularly useful in quality control work or wherever reduction of calculation time in computing sample variances is important. Many tables give specific numerical results. *L. A. Aroian.*

Bateman, G. I. The power of the χ^2 index of dispersion test when Neyman's contagious distribution is the alternate hypothesis. Biometrika **37**, 59–63 (1950).

The author generalizes investigations by Hoel [Ann. Math. Statistics **14**, 155–162 (1943); these Rev. **4**, 280] into the sampling distribution of the Poisson index of dispersion z. He supposes the parent population to have Neyman's contagious distribution of type A with two parameters [Ann. Math. Statistics **10**, 35–57 (1939)], the limiting case in which this reduces to the Poisson distribution was treated by Hoel. The author gives approximate expressions, too elaborate for reproduction here, for the kth order moment ($k=1, 2, 3, 4$) of z. These are stated to be correct to terms of order N^{k-3} when the sample size N is large and to terms of order $(\mu_1')^{k-4}$ when the population mean μ_1' is large; the magnitude of the omitted remainders is not investigated further. When $\mu_1'\to\infty$ (with N fixed) the author's approximations to the four moments concerned approach the corresponding moments of $\chi^2\mu_2/\mu_1'$, where χ^2 has $N-1$ degrees of freedom and μ_2 is the population variance; the rapidity of approach is illustrated numerically. Using the distribution of $\chi^2\mu_2/\mu_1'$ to approximate to that of z, the author graphs the power described in his title for $1\leqq\mu_2/\mu_1'\leqq 4$ and $N=5$, 10, 20, 50, 100, 200.

H. P. Mulholland (Bath).

Bahadur, Raghu Raj. On a problem in the theory of k populations. Ann. Math. Statistics **21**, 362–375 (1950).

This problem is a generalization of the problem involved in selecting, on the basis of k equal samples from k populations, that population with the largest mean. The selection is here generalized to permit fractions p_i of each population to be taken (a generalization similar to randomized test procedures). Hence a selection function is a k-valued function of the sample (d) such that $s(d)=p_1, p_2, \cdots, p_k$, $\sum_{i=1}^k p_i=1$. Given then a set of distributions $\{G(x,\theta_i)\}$, $i=1, 2, \cdots, k$, with $\omega=\{\theta_1, \theta_2, \cdots, \theta_k\}\in\Omega$, some parameter space, and a criterion function $g(x)$ from which are derived "weight factors" $g_i=\int_{-\infty}^{\infty} g(x)dG(x,\theta_i)$, the problem is to choose $s(d)$ so that $W(\omega, s)=\max_i\{g_i\}-\sum_{i=1}^k g_i p_i(d)$ is minimized for every $\omega\in\Omega$, and W represents the "loss" incurred by choosing the decision $s(d)$ when the true parameter point is ω.

To arrive at a solution the existence of certain statistics relative to the unknown parameters of the $G(x,\theta_i)$ are assumed and also various properties of these statistics. In particular, a certain "ordering" is required for the frequency functions of the statistics from the different populations. Under these conditions and under a condition of "impartiality" on the decision rule [the rule depends only on the order of the statistics], the optimum decision rule is of the form $s(d)=(1, 0, 0, \cdots, 0)$. Some examples are discussed, e.g., the problem of the greatest mean of k normal populations. The optimum decision rule here is: "always select only the population corresponding to the greatest $\bar{x}_i$."

D. G. Chapman (Seattle, Wash.).

Massey, F. J., Jr. A note on the power of a non-parametric test. Ann. Math. Statistics **21**, 440–443 (1950).

Let $S_n(x)$ be the sample cumulative distribution function of n independent observations on a continuous chance variable X. Consider the test of the hypothesis that the unknown cumulative distribution function of X equals $F_0(x)$ based on the statistic $\max_x|F_0(x)-S_n(x)|$, large values being critical. It is shown that the test is consistent against any continuous alternative $F_1(x)\not\equiv F_0(x)$, though possibly biased for finite n. A lower bound for the power of the test is found, and an indication is given how these results may be extended to the corresponding two sample test.

G. E. Noether (New York, N. Y.).

Theil, H. A rank-invariant method of linear and polynomial regression analysis. I. Nederl. Akad. Wetensch., Proc. **53**, 386–392 = Indagationes Math. **12**, 85–91 (1950).

Given the linear regression $\vartheta_i=a_0+a_1\xi_i$, $i=1, 2, \cdots, n$, it is assumed that the observed value x_i of the independent value differs from ξ_i by u_i, that the true value η_i of the dependent variable differs from the regression value ϑ_i by w_i, and that the observed value y_i of the dependent variable differs from the true value η_i by v_i. Regarding u_i, v_i, w_i as random variables with the n triples (u_i, v_i, w_i) stochastically independent, the author develops nonparametric confidence intervals for a_0 and a_1. It is assumed that the errors u_i are bounded in such a way that the observed ordering by increasing magnitude of the x_i is the same as that for the ξ_i. Then with the x_i's so ordered and with $n_1=\frac{1}{2}n$ (if n is odd, the middle observation is dropped), the difference quotients $\Delta(i, n_1+i)=(y_{n_1+i}-y_i)/(x_{n_1+i}-x_i)$ are also ranked. Sufficient conditions are imposed that the Δ_j (Δ_j is the jth Δ in rank order) have the median a_1 and from this a confidence interval for a_1 is found in the form $(\Delta_r\leqq a_1\leqq\Delta_{n_1-r+1})$ with a coefficient determined from an incomplete β-function with argument $\frac{1}{2}$. By a further extension of this general method simultaneous confidence intervals for a_0 and a_1 are found. These methods are called incomplete since only part of the difference quotients $(y_i-y_j)/(x_i-x_j)$ are used. A complete method using all of these quantities in conjunction with Kendall's rank correlation coefficient is also devised to give a confidence interval for a_1. Finally a nonparametric test for linearity is indicated. *C. C. Craig.*

Theil, H. A rank-invariant method of linear and polynomial regression analysis. II. Nederl. Akad. Wetensch., Proc. **53**, 521–525 = Indagationes Math. **12**, 173–177 (1950).

This paper is the second part of the paper reviewed above. Here the methods developed in the first part for determining confidence intervals for parameters in linear regression equations in two variables are directly and quite analogously extended to cases of three or more variables under conditions also quite analogous. *C. C. Craig.*

Quenouille, M. H. The joint distribution of serial correlation coefficients. Ann. Math. Statistics **20**, 561–571 (1949).

The joint distribution of circularly defined serial correlation coefficients is derived on the assumptions that the observations are normally and independently distributed. The results may be extended to the case where the observations satisfy the relation $a_0x_i+a_1x_{i-1}+\cdots+a_mx_{i-m}=\epsilon_i$, $\epsilon_{n+i}=\epsilon_i$ and the ϵ's are normally and independently distributed. The distributions are difficult to compute and approximate forms are investigated. *H. Chernoff.*

Ghurye, S. G. A method of estimating the parameters of an autoregressive time-series. Biometrika **37**, 173–178 (1950).

In the absence of a general and easily applicable method, it is usual to estimate the parameters in the autoregressive scheme $f(t)\equiv U_t+\alpha U_{t-1}+\beta U_{t-2}=\epsilon_t$ (ϵ being a random disturbance function) by use of the first two or three observed serial correlations. The author proposes a method which uses a larger number of serial correlations; his condition for the estimation of the parameters is that a weighted sum of squares of serial correlations of the values of f be minimized. The sum used commences with the third serial correlation, which is the first one whose expected value is zero, and is extended to any moderate order substantially less than the length of the given sequence. The author tests his method on two artificial series, with quite favorable results. *A. Blake* (Aurora, Ill.).

Salvemini, Tommaso. Contributo allo studio della mutabilità delle serie cicliche. Metron **15**, 29–69 (1949).

The author investigates cyclical time series and discusses several descriptive indices introduced by C. Gini [Atti Ist. Veneto Sci. Lett. Arti. Parte I. **77**, 397–461 (1918); republished in Memorie di metodologia statistica, Giuffrè, Milano, 1939, pp. 411–472] and by the author [Atti Secondo Congresso Un. Mat. Ital. Bologna, 1940, pp. 657–671 (1942); these Rev. **8**, 474]. He criticizes these indices and proposes two new statistics which are closely related to them. The paper belongs in the field of descriptive statistics; no attempt is made to discuss problems of statistical inference or to investigate the distribution of the indices under discussion. *E. Lukacs* (Washington, D. C.).

Kotzig, Anton. The weights of the results of partial tests for determining the total result of the test. Aktuárské Vědy **8**, 129–137 (1949).

Black, A. N. Weighted probits and their use. Biometrika **37**, 158–167 (1950).

A detailed computational scheme for probit analysis is outlined. The method uses tables which are included in the paper and an example is worked out. *H. Chernoff.*

TOPOLOGY

Smith, C. A. B., and Tutte, W. T. A class of self-dual maps. Canadian J. Math. **2**, 179–196 (1950).

A "map" M is a dissection of the 2-sphere S^2 into a finite cell-complex. A flow of electricity through the 1-cells of M, regarded as wires of unit resistance, is known to correspond to a dissection of a rectangle into squares [cf. Brooks, Smith, Stone, Tutte, Duke Math. J. **7**, 312–340 (1940); these Rev. **2**, 153]. The corresponding flow for the dual map M^*, and the relation between the flow and the rectangle dissection, are derived afresh, but with emphasis on "c-nets" rather than on "p-nets," and M is "self-dual" if it is combinatorially isomorphic to M^*; there then exists a periodic homeomorphism ϕ of S^2 onto itself which transforms M into M^*. In particular, if ϕ has period 2 and reverses orientation (and so is equivalent to a reflection in a plane or point), M is a (planar or central) "reflex." Factorization theorems are obtained for the currents (suitably normalized) in the flows for reflexes, and it is shown how central reflexes can be used to derive dissections of rectangles into unequal squares whose sides can be expressed by relatively small integers; several examples are given. Planar reflexes give rise to diagonally symmetric dissections of a square into smaller squares. *A. H. Stone.*

Tutte, W. T. Squaring the square. Canadian J. Math. **2**, 197–209 (1950).

[Cf. the preceding review and reference given there.] This paper gives methods for modifying planar reflexes so as to obtain dissections of a square into finitely many unequal smaller squares, and it is further shown how the dissections can be made "simple" (containing no dissection of a smaller rectangle) and "uncrossed" (no point belonging to four of the small squares). One method depends upon a criterion under which part of an electric network may be altered without altering the flow in the remainder; another uses planar reflexes which have further symmetry; and a third uses wires of "negative resistance" (corresponding to overlapping squares in the dissection). This method yields incidentally a (presumably infinite) family of "simple" dissections of a rectangle of sides 15:17 into unequal squares. *A. H. Stone* (Manchester).

Miller, Harlan C. On unicoherent continua. Trans. Amer. Math. Soc. **69**, 179–194 (1950).

In this paper a study is made of hereditary unicoherence of continua in connection with hereditary indecomposability and also the property of being atriodic. Among other results it is shown, for example, that in order for a hereditarily decomposable continuum to be hereditarily unicoherent it is necessary and sufficient that it contain no continuum which admits an upper semi-continuous decomposition into continua with a simple closed curve as hyperspace. Also a group of theorems is proved showing analogy in structure between hereditarily decomposable hereditarily unicoherent continua and acyclic continuous curves. *G. T. Whyburn.*

Balanzat, Manuel. On α-regular écartized spaces. Revista Unión Mat. Argentina **14**, 90–98 (1949). (Spanish)

The author shows by an example that the condition of α-regularity for a space endowed with a (not necessarily symmetric) écart is actually more general than that of regularity [in the terminology of M. Fréchet, Portugaliae Math. **5**, 121–131 (1946); these Rev. **8**, 48]. Examples are also given to show that, for a nonmetrizable α-regular space with a numerical écart, the requirement of a set being compact, or separable, is not as drastic as it is known to be in the case of a space with an écart which admits no topologically equivalent numerical écart. *L. Nachbin.*

Nagata, Jun-iti. A characterization of the uniform topology of a uniform space by the lattice of its uniformity. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. 1, 39–45 (1950).

Given uniform spaces R, S whose uniformities are the families of coverings $\mathfrak{M}_x$, $x\in\mathfrak{X}$ and $\mathfrak{N}_y$, $y\in\mathfrak{Y}$ respectively [J. W. Tukey, Convergence and Uniformity in Topology, Princeton University Press, 1940; these Rev. 2, 67], this note states conditions for the uniform homeomorphism of R and S in terms of isomorphism between the lattices $\mathfrak{M}_x$ and $\mathfrak{N}_y$. The results are: (1) R and S are uniformly homeomorphic if and only if there is an isomorphism between the lattices $\mathfrak{M}_x$ and $\mathfrak{N}_y$ which preserves the relations of refinement and Δ-refinement [loc. cit., pp. 43–45]; (2) if R has no isolated point, then the existence of any lattice isomorphism between $\mathfrak{M}_x$ and $\mathfrak{N}_y$ is sufficient for the existence of a uniform isomorphism between R and S. *L. W. Cohen.*

Katětov, Miroslav. On rings of continuous functions and the dimension of compact spaces. Časopis Pěst. Mat. Fys. 75, 1–16 (1950). (Russian. English and Czech summaries)

In this paper a contribution is made to the theory of rings of continuous real-valued functions. Let P be a completely regular space. Let ind P be the Menger-Urysohn dimension of P. A mapping Φ of P into a set Q is called light in the present paper if ind $\Phi^{-1}(q)\leqq 0$ for all $q\in Q$. For every cardinal number $\mathfrak{m}>0$, let $E^{\mathfrak{m}}$ be the Cartesian product of $\mathfrak{m}$ replicas of the space E of real numbers, and let E^0 be a space consisting of exactly one point. Let $C(P)$ be the ring of all bounded, real-valued, continuous functions defined on P, with point-wise addition and multiplication. A subring C_1 of $C(P)$ is said to be algebraically closed if $\{f_0, f_1, \cdots, f_n\}\subset C_1$, $\varphi\in C(P)$, and $\sum_{i=0}^{n} f_i\varphi^i=0$ imply $\varphi\in C_1$. A subring of $C(P)$ which contains all constant functions, is algebraically closed, and is closed under uniform convergence, is said to be analytically closed. A subset M of $C(P)$ (which may be void) is said to be an analytic basis for $C(P)$ if the smallest analytically closed subring of $C(P)$ containing the set M is the ring $C(P)$ itself. The least cardinal number of an analytic basis for $C(P)$ is called the analytic dimension of $C(P)$ and is denoted by the symbol dim $C(P)$. For P a compact Hausdorff space, it is proved that for every analytically closed subring of $C(P)$, there exists a dissection of P into pair-wise disjoint, closed, connected sets A_λ (λ running through some index class Λ) such that C_1 is the set of all continuous real-valued functions which are constant on each set A_λ. Conversely, for P any completely regular space, any such subring of $C(P)$ is analytically closed. Using these two results as well as a number of topological remarks, the author proves the following facts. (I) For a nonvoid compact Hausdorff space P, let $\mathfrak{m}$ be the least cardinal number such that P admits a light continuous mapping into $E^{\mathfrak{m}}$. If $\mathfrak{m}<\aleph_0$, then ind $P=\mathfrak{m}$; if $\mathfrak{m}=\aleph_0$, then ind $P=\infty$. If $\mathfrak{m}>\aleph_0$, then ind P can have any value >0. (II) For P a nonvoid compact Hausdorff space, dim $C(P)$ is the least cardinal number $\mathfrak{m}$ such that P admits a light continuous mapping into $E^{\mathfrak{m}}$. (III) Let P be a nonvoid compact Hausdorff space. If dim $C(P)<\aleph_0$, then dim $C(P)=$ ind P. If dim $C(P)=\aleph_0$, then ind $P=\infty$. *E. Hewitt* (Seattle, Wash.).

Katětov, Miroslav. A theorem on the Lebesgue dimension. Časopis Pěst. Mat. Fys. 75, 79–87 (1950). (English. Czech summary)

The notation and terminology in this review are those of the preceding review. Let P be any normal space, and let dim P be the Lebesgue dimension of P. For an arbitrary completely regular space P, let dim $P=$dim βP, where βP is the Stone-Čech compactification of P. It is proved that for normal P, βP and P have the same Lebesgue dimension, so that the definition of dim P as dim βP is consistent. Let the analytic pseudo-dimension of the ring $C(P)$ be the least cardinal number $\mathfrak{m}$ such that every countable set $M\subset C(P)$ is contained in an analytically closed subring of $C(P)$ having an analytic basis of $\mathfrak{m}$ or fewer elements. Then, for every completely regular space P, dim P is equal to the analytic pseudo-dimension of $C(P)$. Various results are also obtained concerning the dimension function dim P. *E. Hewitt.*

Gale, David. Compact sets of functions and function rings. Proc. Amer. Math. Soc. 1, 303–308 (1950).

The author generalizes the reviewer's characterization of compact sets of mappings of a space X (either locally compact or satisfying the first countability axiom) into a metric space Y [see Ann. of Math. (2) 47, 496–502 (1946); these Rev. 8, 165]. Let X be a k-space, that is a Hausdorff space in which $S\subset X$ is closed if and only if S intersects every compact subset of X in a compact subset; the class of k-spaces includes all locally compact and all first countable spaces. Let Y be a regular Hausdorff space, and let Y^X be the space of continuous mappings of X into Y, provided with the compact-open topology. The author shows that $F\subset Y^X$ is compact if and only if (1) F is closed in Y^X, (2) $F(x)$ is compact for each $x\in X$, (3) if Q is closed in F and C is closed in Y, then $\bigcup Q^{-1}(C)$ is closed in X. Condition (3) takes the place of equicontinuity of F, which was used when Y is metric. As an application the following result is proved. Let X be a k-space, $A(X)$ a sub-algebra of the algebra of continuous real functions on X which separates points from closed sets in X, and let H be the set of continuous homomorphisms of $A(X)$ into the reals; then the mapping $\phi(X)\subset H$ defined by $\phi(x)(f)=f(x)$ for all $f\in A(X)$ is a homeomorphism of X onto H. *S. B. Myers.*

Brodskiĭ, M. S. On special ϵ-nets. Uspehi Matem. Nauk (N.S.) 5, no. 2(36), 191–195 (1950). (Russian)

The author proves several theorems about compacta by means of ϵ-nets with suitable extra properties. Theorem 1. Let Q be a compactum, Δ a positive number, and s a continuous mapping of Q onto itself such that $d(x, y)<\Delta$ implies $d(sx, sy)\leqq d(x, y)$. Then $d(x, y)<\Delta$ implies $d(sx, sy)=d(x, y)$, and also $d(x, y)\geqq\Delta$ implies $d(sx, sy)\geqq\Delta$. Theorem 2. If the mappings of a semi-group G of mappings of a compactum Q onto itself are equi-continuous, there exists a mean on the real continuous functions on Q (that is, a nonnegative linear functional of norm 1) invariant under G. Theorem 3. In each compact, convex subset Q of a normed linear space there exists a point invariant under all affine isometries of Q on itself. Most of the novelty is in the comparative simplicity of the proofs; for example, Theorem 3 is a special case of a result of Brodskiĭ and Mil'man [Doklady Akad. Nauk SSSR (N.S.) 59, 837–840 (1948); these Rev. 9, 448]. To prove Theorem 1, for each Δ_1, $0<\Delta_1<\Delta$, take $0<\epsilon<\Delta_1$; let M be the set of all ϵ-nets with minimal number of elements n. Then let M_1 be the subset of M consisting of ϵ-nets with a maximal number of pairs x_i, x_j such that $d(x_i, x_j)\leqq\Delta_1$. Since M_1 is compact, the function $\sum d(x_i, x_j)$, where the sum is extended only over pairs with $d(x_i, x_j)\leqq\Delta_1$, attains its minimum on a nonempty subset M_2 of M_1, and S carries each ϵ-net of M_2 into another ϵ-net of M_2 so that $d(sx_i, sx_j)=d(x_i, x_j)$ when $d(x_i, x_j)\leqq\Delta_1$. Letting Δ_1 approach Δ and ϵ approach 0 completes the proof. *M. M. Day.*

Scorza Dragoni, G. Alcuni teoremi sulle traslazioni piane generalizzate. Ann. Triestini. Sez. 2 (4) 1(17), 5–40 (1947).

Let t be an orientation-preserving fixed-point-free autohomeomorphism of the plane. A translation segment of t is a linear segment $\lambda=PQ$ such that $Q=t(P)=\lambda\cap t(\lambda)$. A translation segment may possess a positive (or negative) characteristic line or segment. A line l is a positive characteristic line of λ if it meets the trajectory

$$\sigma(\lambda)=\cdots+t^{-1}(\lambda)+\lambda+t(\lambda)+\cdots$$

in a single point, an interval point of λ, and if the ray of l which lies on the positive side of $\sigma(\lambda)$ fails to meet its image under t. The definition of characteristic segments is somewhat more complicated. Now let W be a collection of translation segments of t whose initial points form a bounded set. The main theorem asserts that W is the union of a finite number of disjoint sets $W_1, \cdots, W_n$ such that the translation segments in W_i, $i=1, \cdots, n$, have a common positive characteristic element (arc or segment) and a common negative characteristic element. *P. A. Smith.*

Reidemeister, Kurt. Complexes and homotopy chains. Bull. Amer. Math. Soc. **56**, 297–307 (1950).

This is an expository paper. The author first outlines his theory of homotopy chain groups, and then discusses a number of theorems which can be proved by means of this machinery. These include Hopf's theorem concerning the influence of the fundamental group on the second homology group, the author's own theorem that the fundamental group of a three-dimensional manifold, if Abelian, must be cyclic or the free group on three generators, and the combinatorial classification of lens spaces. Some of the proofs are outlined.

E. G. Begle (New Haven, Conn.).

Olum, Paul. Obstructions to extensions and homotopies. Ann. of Math. (2) **52**, 1–50 (1950).

The extensions and homotopies of maps of a connected locally finite polyhedron P into an arcwise connected topological space Y is studied in the most general form. By means of Steenrod's notion of a system of local groups, the author is able to generalize the usual obstruction methods of Eilenberg without assuming the n-simplicity of the space Y. The formulations as well as the proofs are much complicated by the unnecessary use of the simplicial maps of the (semi-simplicial) singular complex.

For the extension problem, let P' be a closed subpolyhedron of P and $x_0\in P'$. Consider a given map $f': P'\to Y$ and a given homomorphism θ of $\pi_1(P, x_0)$ into $\pi_1(Y, f'(x_0))$. Then f' induces a homomorphism θ' of $\pi_1(P', x_0)$ into $\pi_1(Y, f'(x_0))$, and f' is extendable over $P^2\cup P'$ so as to induce θ if and only if $\theta'=\theta\xi$ where ξ is the injection of $\pi_1(P', x_0)$ into $\pi_1(P, x_0)$. If f' is extendable over $P^{n-1}\cup P'$, where $n\geqq 3$, so as to induce θ, then a nonvoid obstruction set $O_\theta^n(f')$ of the cohomology group $H^n(P, P', \theta^*\pi_{n-1})$ is defined. The main extension theorem states that f' is extendable over $P^n\cup P'$ so as to induce θ if and only if $O_\theta^n(f')$ contains the zero element. This generalizes the "fundamental extension lemma" of the reviewer [Trans. Amer. Math. Soc. **64**, 336–358 (1948); these Rev. **10**, 138] inasmuch as Y need not be $(n-1)$-simple. For the homotopy problem, generalizations of the reviewer's "fundamental homotopy lemma" are established analogously.

S. T. Hu (Princeton, N. J.).

Chang, Su-Cheng. Homotopy invariants and continuous mappings. Proc. Roy. Soc. London. Ser. A. **202**, 253–263 (1950).

A finite, connected polyhedron K is called an A_n^2-polyhedron if dimension $K\leqq n+2$, and $\pi_r(K)=0, r=1, \cdots, n-1$. J. H. C. Whitehead [Ann. Soc. Polon. Math. **21** (1948), 176–186 (1949); these Rev. **11**, 48] has reduced the problem of classifying A_n^2-polyhedra according to homotopy type to a purely algebraic problem in case $n\geqq 2$. However, he left this algebraic problem unsolved. The author now gives a complete solution to this problem for $n>2$; the case $n=2$ offers great difficulties. He defines certain new numerical invariants of A_n^2-polyhedra, called the secondary torsions. The secondary torsions are effectively calculable in a finite number of steps. Their definition involves the squaring operation of Steenrod [Ann. of Math. (2) **48**, 290–320 (1947); these Rev. **9**, 154]. The main result is that the homotopy type of an A_n^2-polyhedron, $n>2$, is completely determined by its Betti numbers, torsion coefficients, and secondary torsions.

The author proves several corollaries of this result. (1) He defines seven different kinds of "elementary" A_n^2-polyhedra, which have a particularly simple structure, and proves that every A_n^2-polyhedron is of the same homotopy type as a "normal" A_n^2-polyhedron, which consists of the union of a finite number of elementary A_n^2-polyhedra with a single point in common. (2) If K is an A_n^2-polyhedron, $n>2$, then the homotopy group $\pi_{n+1}(K)$ and the cohomotopy group $\pi^{n+1}(K)$ may be effectively calculated, provided the Betti numbers, torsion coefficients, and secondary torsions are known. *W. S. Massey* (Providence, R. I.).

Borel, Armand, et Serre, Jean-Pierre. Impossibilité de fibrer un espace euclidien par des fibres compactes. C. R. Acad. Sci. Paris **230**, 2258–2260 (1950).

The authors prove the statement given in the title by applying Leray's theory of fiber maps. One sees easily that one can assume the fiber to be either discrete, in which case the result follows from a theorem of P. A. Smith, or arcwise connected. In the second case one considers the Leray-Koszul sequence which begins with $H(F)\otimes H(B)$, and ends with $H(R^n)$, where H refers to the singular cohomology ring with compact supports, R^n is Euclidean n-space, F the fiber, and B the base. One concludes that $H(B)$ is not 0; and then, starting from $1\otimes y$, where 1 is the unit of $H(F)$, and y a nonzero element of $H(B)$ of lowest possible dimension p ($\leqq n-1$, since B is at most $(n-1)$-dimensional), one would obtain nonzero elements in $H^p(R^n)$, a contradiction. The theorem remains true if R^n is replaced by a simply connected manifold with the same cohomology ring.

H. Samelson (Ann Arbor, Mich.).

Alexandroff, P. Die grundlegenden Dualitätssätze für nichtabgeschlossene Mengen des n-dimensionalen Raumes. Sowjetwissenschaft **1948**, no. 1, 176–243 (1948).

Translated from Mat. Sbornik N.S. **21(63)**, 161–232 (1947); these Rev. **9**, 456.

Koszul, Jean-Louis. Homologie et cohomologie des algèbres de Lie. Bull. Soc. Math. France **78**, 65–127 (1950).

Making use of the theorems of de Rham, it is possible to reduce the study of the homology and cohomology (with real coefficients) of a compact connected Lie group G to purely algebraic problems on the Lie algebra $\mathfrak{a}$ of G. In particular, the cohomology algebra of G becomes identical

to an algebra which may be defined in terms of $\mathfrak{a}$ alone, viz., the cohomology algebra $H^*(\mathfrak{a})$ of $\mathfrak{a}$. But the algebraic definition of $H^*(\mathfrak{a})$ does not require $\mathfrak{a}$ to be the Lie algebra of a Lie group: $\mathfrak{a}$ may be any Lie algebra over any basic field. It then becomes an interesting question to find out to which extent the known cohomological properties of compact groups extend to similar properties of the cohomology algebras of arbitrary Lie algebras. The first part of the present paper is concerned with such questions.

Let $\mathfrak{a}$ be a Lie algebra over a field of characteristic $\neq 2$. Let $\Lambda(\mathfrak{a})$ be the exterior algebra of $\mathfrak{a}$ (considered as a vector space) and $\Lambda^*(\mathfrak{a})$ that of the dual space of $\mathfrak{a}$. The author defines linear operators ∂ in $\Lambda(\mathfrak{a})$ and δ in $\Lambda^*(\mathfrak{a})$ which have the familiar properties of boundary and coboundary operators: $\partial^2=0$, $\delta^2=0$, ∂ lowers the degree by 1, δ raises it by 1. These operators define, as usual, a homology space $H(\mathfrak{a})$ and a cohomology space $H^*(\mathfrak{a})$; these vector spaces are canonically dual to each other. Moreover, it turns out that δ is an antiderivation; i.e., $\delta(xy)=\delta(x)y+(-1)^r x\delta(y)$, if x is homogeneous of degree r. This circumstance permits the introduction of a multiplication in $H^*(\mathfrak{a})$ which makes it an algebra. For any $p\geqq 0$, the dimension of the space of elements of degree p in $H(\mathfrak{a})$ or in $H^*(\mathfrak{a})$ is called the pth Betti number $B_p(\mathfrak{a})$ of $\mathfrak{a}$; this number is 0 if $p>n$, where n is the dimension of $\mathfrak{a}$.

The first question to be considered concerns the validity for an arbitrary $\mathfrak{a}$ of the Poincaré duality theorem, i.e., of the relation $B_p(\mathfrak{a})=B_{n-p}(\mathfrak{a})$. The condition for this to be true is shown to be that $\mathfrak{a}$ be unimodular, i.e., that, for every $X\in\mathfrak{a}$, the trace of the linear mapping $Y\to[X, Y]$ be 0.

Next, a generalization is sought of Hopf's theorem on the structure of the cohomology of a compact Lie group [E. Hopf, Ann. of Math. (2) **42**, 22–52 (1941); these Rev. **3**, 61]. To every X in $\mathfrak{a}$ there is associated a derivation $\theta(X)$ of $\Lambda(\mathfrak{a})$ which extends the adjoint operation of X, and, by duality, a derivation $\theta^*(X)$ of $\Lambda^*(\mathfrak{a})$. Those elements of $\Lambda(\mathfrak{a})$ (or of $\Lambda^*(\mathfrak{a})$) which are mapped upon 0 by all operations $\theta(X)$ (or $\theta^*(X)$) are called invariant. The Lie algebra $\mathfrak{a}$ is said to satisfy condition (P) if every homology class of $\mathfrak{a}$ contains at least one invariant element. This implies that $\mathfrak{a}$ is unimodular and that every cohomology class of $\mathfrak{a}$ contains some invariant element. Moreover, in that case a multiplication can be defined in a natural manner in the homology space $H(\mathfrak{a})$, which turns this space into an algebra; if $\mathfrak{a}$ is the Lie algebra of a compact connected Lie group G, then the multiplication defined in this manner is identical to the Pontryagin multiplication of homology classes of G. If the Lie algebras $\mathfrak{b}$ and $\mathfrak{a}$ both satisfy condition (P), then the linear mapping of $H(\mathfrak{b})$ into $H(\mathfrak{a})$ which corresponds to any given homomorphism of $\mathfrak{b}$ into $\mathfrak{a}$ is proved to be a homomorphism of the algebra $H(\mathfrak{b})$ into $H(\mathfrak{a})$, a fact which is of decisive importance for the proof of the generalization of Hopf's theorem. Condition (P) is always satisfied if $\mathfrak{a}$ is reductive (i.e., if the representation of $\mathfrak{a}$ given by the operators $\theta(X)$ is completely reducible). If the basic field is of characteristic 0, the reductive algebras are the products of Abelian and semi-simple algebras. If $\mathfrak{a}$ is reductive, then every homology (or cohomology) class contains exactly one invariant element. Let now $\mathfrak{a}$ be a reductive algebra over a field of characteristic 0. A primitive element of $H^*(\mathfrak{a})$ is defined to be an element which is orthogonal (in the duality between $H(\mathfrak{a})$ and $H^*(\mathfrak{a})$) to the elements of degree 0 and to all products of two elements of degrees >0 of $H(\mathfrak{a})$; the primitive elements of $H(\mathfrak{a})$ are defined in a similar manner. Let $P(\mathfrak{a})$ and $P^*(\mathfrak{a})$ be the spaces of primitive elements for the homology and the cohomology. Then it is proved that the homogeneous primitive elements are of odd degrees, and that $H(\mathfrak{a})$ and $H^*(\mathfrak{a})$ may be identified to the exterior algebras on $P(\mathfrak{a})$ and $P^*(\mathfrak{a})$ respectively; this is the generalization of Hopf's theorem. Moreover, this identification is such that the duality between homology and cohomology becomes identical to the canonical extension to the exterior algebras of the duality between $P(\mathfrak{a})$ and $P^*(\mathfrak{a})$ which is induced by the duality between $H(\mathfrak{a})$ and $H^*(\mathfrak{a})$. The proof of this fundamental theorem is based on the study of the homomorphism of $\mathfrak{a}$ into $\mathfrak{a}\times\mathfrak{a}$ which maps any X in $\mathfrak{a}$ upon (X, X); this homomorphism is an algebraic counterpart to the mapping $(s, t)\to st$ of $G\times G$ into G, where G is a group; Hopf's proof was based on the study of this mapping.

It is then shown that, for a semi-simple $\mathfrak{a}$, the third Betti number is equal to the number of linearly independent symmetric bilinear forms on $\mathfrak{a}\times\mathfrak{a}$ (this connexion between alternating invariants and symmetric invariants has been later extended by A. Weil to the primitive elements of any dimension). It follows in particular that the Betti number of a compact simple connected Lie group is 1.

The second part of the paper is concerned with the relative cohomology algebra $H^*(\mathfrak{a}; \mathfrak{b})$ of a Lie algebra $\mathfrak{a}$ with respect to a subalgebra $\mathfrak{b}$. A cochain γ of $\mathfrak{a}$ is called orthogonal to $\mathfrak{b}$ if it is in the algebra generated by those linear functions on $\mathfrak{a}$ which vanish on $\mathfrak{b}$; γ is called $\mathfrak{b}$-invariant if it is mapped upon 0 by the operators $\theta^*(X)$ for all X in $\mathfrak{b}$. Those cochains which are at the same time orthogonal to $\mathfrak{b}$ and $\mathfrak{b}$-invariant are called relative cochains. They form an algebra which is mapped into itself by the operator δ and which therefore has a cohomology algebra $H^*(\mathfrak{a}; \mathfrak{b})$. If $\mathfrak{a}$ and $\mathfrak{b}$ are the Lie algebras of a connected compact Lie group G and of a closed connected subgroup H of G, then $H^*(\mathfrak{a}; \mathfrak{b})$ is the cohomology algebra of the homogeneous space G/H.

The subalgebra $\mathfrak{b}$ is called reductive in $\mathfrak{a}$ if the representation of $\mathfrak{b}$ given by the operators $\theta(X)$, $X\in\mathfrak{b}$, operating on $\Lambda(\mathfrak{a})$, is completely reducible. If this is the case and $\mathfrak{a}$ is moreover unimodular, then it is proved that the Poincaré duality theorem holds in $H^*(\mathfrak{a}; \mathfrak{b})$. The most interesting results concern the case where $\mathfrak{a}$ is reductive and $\mathfrak{b}$ reductive in $\mathfrak{a}$. The identity mapping of $\mathfrak{b}$ into $\mathfrak{a}$ defines a homomorphism φ^* of $\Lambda^*(\mathfrak{a})$ into $\Lambda^*(\mathfrak{b})$ and therefore also a homomorphism $\bar{\varphi}^*$ of $H^*(\mathfrak{a})$ into $H^*(\mathfrak{b})$. Let m be the dimension of $\mathfrak{b}$; if some element $\neq 0$ of degree m of $H^*(\mathfrak{b})$ appears in $\bar{\varphi}^*(H^*(\mathfrak{a}))$, then it is proved that the image of $H^*(\mathfrak{a})$ is the whole of $H^*(\mathfrak{b})$ and that $H^*(\mathfrak{a})$ is isomorphic with the tensor product $H^*(\mathfrak{b})\otimes H^*(\mathfrak{a}; \mathfrak{b})$ (the definition of the multiplication in this tensor product being slightly modified in relation with the gradations in the factors). This case corresponds to the case of a closed subgroup not homologous to 0 of a compact Lie group; the results we have mentioned are therefore a generalization of a theorem of Samelson [Ann. of Math. (2) **42**, 1091–1137 (1941), Satz VI; these Rev. **3**, 143].

The author also obtains results in the case where $\mathfrak{b}$ is not assumed to be nonhomologous to 0. Let c be an element of $H^*(\mathfrak{b})$; in order for c to belong to the image of $H^*(\mathfrak{a})$, it is necessary and sufficient that there should exist a cocycle γ of $\mathfrak{a}$ such that $\varphi^*(\gamma)$ is a representative of the class c. Now, the element c is called transgressive if there exists a cochain γ of $\mathfrak{a}$ such that $\varphi^*(\gamma)$ is a representative of c and which has furthermore the property that $\delta\gamma$ is a relative cochain of $\mathfrak{a}$ with respect to $\mathfrak{b}$. It is then proved that every primitive element of $H^*(\mathfrak{b})$ is transgressive.

Moreover, if γ is such that $\delta\gamma$ is a relative cochain, then it is a relative cocycle and it yields an element of the relative cohomology algebra $H^*(\mathfrak{a};\mathfrak{b})$. The identity mapping of the algebra of relative cochains into $\Lambda^*(\mathfrak{a})$ gives rise to a homomorphism ψ of $H^*(\mathfrak{a};\mathfrak{b})$ into $H^*(\mathfrak{a})$, and it is clear that the image by ψ of the cohomology class of an element of the form $\delta\gamma$ is 0. The author proves that, if for a certain dimension $p-1$ there are r primitive elements of $H^*(\mathfrak{b})$ which are linearly independent modulo the image of $H^*(\mathfrak{a})$, then these elements yield (in the manner indicated above) r linearly independent elements of degree p of $H^*(\mathfrak{a};\mathfrak{b})$ which are mapped upon 0 by ψ; and, more precisely, any set of homogeneous generators of $H^*(\mathfrak{a};\mathfrak{b})$ must contain at least r elements of degree p. It should be observed here that p is necessarily even.

The proof makes use of an algorithm introduced in topology by Leray [C. R. Acad. Sci. Paris **222**, 1366–1368, 1419–1422 (1946); these Rev. **8**, 49], which appears here as follows. A sequence of differential graded algebras E_s $(s=0, 1, \cdots)$ is constructed in which each term is the cohomology algebra of the previous one (each E_s has a differential operator Δ_s such that $\Delta_s^2=0$; Δ_s raises the degree by s units). The terms of this sequence become ultimately identical with $H^*(\mathfrak{a})$; a result established by the author is that E_2 is isomorphic to the tensor product $H^*(\mathfrak{b})\otimes H^*(\mathfrak{a};\mathfrak{b})$. If $\mathfrak{b}$ is not homologous to 0 in $\mathfrak{a}$, then all terms of the sequence from the second one on are identical with each other.

C. Chevalley (New York, N. Y.).

Iwamoto, Hideyuki. On integral invariants and Betti numbers of symmetric Riemannian manifolds. II. J. Math. Soc. Japan **1**, 235–243 (1949).

[For part I of this paper and for notation cf. the same vol., 91–110 (1949); these Rev. **11**, 377.] The author describes the invariant differential forms and the cycles of the manifolds $S(n)$, $C(n)$, $A^{\pm}(n)$, $\bar{A}^{\pm}(n)$. The description is based on a different interpretation of these manifolds. It is shown, e.g., that $S(n)$ can be interpreted as the space of isotropic m-planes in P_n $(n=2m)$; if $S\in S(n)$, then the vectors $x+(-1)^iSx$, $x\in R_n$, form the corresponding isotropic plane. If $x_1, \cdots, x_m$ form a unitary basis in such a plane, then $\omega_{ij}=(x_i, dx_j)$ are differential forms of $S(n)$; the products $\Omega_s=\omega_{i_1i_2}\cdot\omega_{i_3i_4}\cdots\omega_{i_{2s}i_1}$, $s=1, 2, \cdots, m-1$, form a system of generators for the invariant differential forms. The cycles of $S(n)$ can be interpreted as certain Schubert varieties in the space of isotropic planes. The discussion for the other manifolds is similar. Corresponding constructions for the Grassmann manifolds $A(n, k)$, $S(n, k)$, $R(n, k)$ are briefly indicated.

H. Samelson (Ann Arbor, Mich.).

GEOMETRY

***Jung, Heinrich W. E. Zwei merkwürdige Punkte des Dreiecks.** Hallische Monographien no. 6, p. 3. Max Niemeyer Verlag, Halle (Saale), 1948. RM 2.20.

Let (T) be the triangle having for vertices the zeros of a polynomial of the third degree $g(z)$ of a complex variable z. The author shows that the ellipse (E) touching the sides of (T) at their mid-points has for its foci the two "remarkable points" of (T) determined by the zeros of the derivative $g'(z)$ of $g(z)$. The relation of (T) to the ellipse (E) and its foci was studied extensively during the last quarter of the 19th century, synthetically and otherwise [see, for instance, J. Casey, A Treatise on the Analytic Geometry of the Point, Line, Circle, and Conic sections, 2d ed., Longmans, Green & Co., London, 1893, pp. 451–458]. For bibliographical references concerning the relation of the zeros of $g(z)$ and $g'(z)$ see Marden [The Geometry of the Zeros of a Polynomial in a Complex Variable, American Mathematical Society, New York, N. Y., 1949, p. 11; these Rev. **11**, 101].

N. A. Court (Norman, Okla.).

Decuyper, M. Sur des triangles en position isogonale. Mathesis **59**, 81–85 (1950).

Two triangles (A), (M) inscribed in the same circle (O), with center O, are said to be in isogonal position if the sum of the three angles which the vectors from O to the vertices of (A) make with a fixed axis through O is equal to the analogous sum for the triangle (M) (mod 2π). It is shown that, if the triangle (A) remains fixed and (M) varies, the altitudes of (M) envelop a tricuspidal hypocycloid of which (O) is the tritangent circle. This result enables the author to give very simple proofs of several known propositions.

N. A. Court (Norman, Okla.).

Goormaghtigh, R. Le pôle biangulaire dans les couples de triangles. Mathesis **59**, 167–172 (1950).

Goormaghtigh, R. Sur l'angle de deux cercles podaires. Mathesis **59**, 99–101 (1950).

Thébault, Victor. Sur les hauteurs du tétraèdre. Ann. Soc. Sci. Bruxelles. Sér. I. **62**, 19–23 (1948).

Bilo, J. Une propriété "orthopolaire" des droites asymptotiques de l'hyperboloïde des hauteurs du tétraèdre. Mathesis **59**, 162–166 (1950).

Lorent, H. Courbes construites à partir d'une conique fixe et d'un faisceau de coniques liées à la donnée. Bull. Soc. Roy. Sci. Liège **19**, 131–140 (1950).

"La présente note traite quelques cas du problème général suivant. Dans le plan XOY, un point $A(x', y')$ parcourt une conique donnée (C_1), sa tangente en A coupe en $B(x, y)$ une autre conique (C_2) appartenant à un faisceau et déterminée par une condition dépendant du point A; on recherche le lieu du point B." *From the author's summary.*

Stoll, A. Die Steinersche Hypozykloide. Elemente der Math. **5**, 73–78 (1950).

The first part of this paper appeared in the same journal **5**, 55–60 (1950); these Rev. **11**, 736.

Fempl, Stanimir. The area of an oblique circular cone. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **4**, 127–134 (1949). (Croatian)

The area in question is expressed in terms of an elliptic integral. *W. Feller* (Princeton, N. J.).

Deaux, R. Sur l'image d'une affinité dans le plan de Gauss. Mathesis **59**, 101–110 (1950).

Let P be the Euclidean plane over the field of complex numbers and let an affinity of P onto itself with real coefficients be given. The author studies the properties of this affinity by employing instead of the rectangular coordinates x and y of a point in P its so-called symmetrical coordinates $x+iy$ and $x-iy$. *W. van der Kulk* (Providence, R. I.).

Löbell, Frank. Eine reelle Deutung der komplexen Vektoren. Math. Z. **52**, 759–769 (1950).

The author represents the lines g of hyperbolic 3-space by the chords of the Neumann sphere, and hence by quadratic equations $g_1\xi^2+g_2\xi+g_3=0$, whose roots represent the "ends" of the lines, or by the corresponding binary forms $g_1\xi^2+g_2\xi\eta+g_3\eta^2$. Two lines g and h have a simultaneous invariant $(g, h)=2g_1h_3-g_2h_2+2g_3h_1$, which vanishes if they intersect at right angles, and a simultaneous covariant

$$[g, h]=(g_1h_2-g_2h_1)\xi^2+2(g_1h_3-g_3h_1)\xi\eta+(g_2h_3-g_3h_2)\eta^2,$$

which represents their common perpendicular. Three lines f, g, h have a simultaneous invariant

$$[f, g, h]=(f, [g, h])=2\det(f_i, g_i, h_i),$$

which vanishes if they have a common perpendicular. If two lines are distant a along their common perpendicular, and lie in planes through this perpendicular inclined at angle α, it is found that $\cos^2(\alpha+ai)=(g, h)^2/(g, g)(h, h)$. In particular, the condition for parallelism is $(g, g)(h, h)=(g, h)^2$. Since the line g is equally well represented by ρg for any nonzero complex number ρ, the symbolism does not exhibit its full perfection till the form g is made to represent a "complex vector" along the line, i.e., a screw comprising rotation α and translation a where $\sin^2(\alpha+ai)=(g, g)$. There is then a striking analogy between the expressions (g, h), $[g, h]$ and the two classical products of vectors in Euclidean space. This is pursued into the realm of kinematics. *H. S. M. Coxeter* (Toronto, Ont.).

Levenberg, Kenneth. A class of non-Desarguesian plane geometries. Amer. Math. Monthly **57**, 381–387 (1950).

The author generalizes the non-Desarguesian geometry of F. R. Moulton, and shows that "Desargues' theorem is one of a large class of theorems which are not valid in each of the non-Desarguesian projective planes" he considers.

L. M. Blumenthal (Columbia, Mo.).

Convex Domains, Extremal Problems, Integral Geometry

Pogorelov, A. V. On regularity of convex surfaces. Uspehi Matem. Nauk (N.S.) **5**, no. 3(37), 188–189 (1950). (Russian)

This paper contains essentially the same results as those in a previous paper [Doklady Akad. Nauk SSSR (N.S.) **67**, 791–794 (1949); these Rev. **11**, 201] with the improvement that $k-2$ and $k-3$ are replaced by $k-1$ and $k-2$ respectively. The proofs, which are only sketched, are based on the following theorem, for which no proof is indicated. Let $F(x, y, z, p, q, r, s, t)=0$ be a partial differential equation of elliptic type. If F has all kth partial derivatives ($k\geqq 3$) with respect to all arguments, then a solution $z(x, y)$ has at least all $(k+1)$st partial derivatives. *H. Busemann.*

Fejes Tóth, László. Ausfüllung eines konvexen Bereiches durch Kreise. Publ. Math. Debrecen **1**, 92–94 (1949).

The author proves the following theorem. Let T be the area of the convex plane region T and let K be the area of its incircle. Suppose that an arbitrary number of nonoverlapping circles of n different sizes are placed in T without overlapping. Then, if t is the area of that part of T which is not covered by the circles,

$$t\geqq T(1-\pi/\sqrt{12})^{n-1}\{1-\max(\pi/\sqrt{12}, K/T)\}.$$

This lower bound is exact, but is attained only in the trivial case where a single circle, namely the incircle, is placed in a region for which $K/T>\pi/\sqrt{12}$. This is deduced from the result, already proved by the author but sketched again here [Comment. Math. Helv. **23**, 342–349 (1949); these Rev. **11**, 455], that, if two or more equal circles are placed in T, their combined area does not exceed $\pi T/\sqrt{12}$, and from a simple lemma which states that if a finite number of circles is removed from T then the density of packing in the remaining region of equal circles of smaller radius than those removed does not exceed $\pi/\sqrt{12}$. *R. A. Rankin.*

Green, John W. Sets subtending a constant angle on a circle. Duke Math. J. **17**, 263–267 (1950).

For a circle C of radius r, a smaller concentric circle of radius $r\sin\frac{1}{2}\alpha$ subtends, in the obvious sense, the same angle α at each point P of C. Again, it is a classical result that an ellipse with semi-axes a and b subtends an angle $\alpha=\frac{1}{2}\pi$ at each point of a concentric circle of radius $(a^2+b^2)^{\frac{1}{2}}$. The question arises as to whether there exist other convex sets K inside C subtending a constant angle on C. [The reviewer has been advised by the author that the latter found, subsequent to the publication of the present paper, that in the special case $\alpha=\pi/2$ the existence of such sets K has been noted by Blaschke, Arch. Math. Phys. (3) **26**, 115–118 (1917).] The author presents a proof that a convex set K inside C which subtends a constant angle $\alpha=\pi-\beta$ at all points P of C is necessarily a concentric circle if β is an irrational multiple of π or if $\beta=(m/n)\pi$ where the rational number m/n in its lowest terms has an even numerator; but there exist noncircular convex sets K for any other angle β between 0 and π, and these sets K can be constructed with a considerable degree of arbitrariness. For the case where such noncircular sets K exist, the author uses a class of generalized convex functions, namely sub-sine functions, to derive several extremal properties involving perimeters, diameters, and widths. *E. F. Beckenbach.*

Koschmieder, Lothar. Äusserstwerte an Lamëschen Kurven und Flächen. Gaceta Mat. (1) **1**, 53–59 (1949).

Let C: $(x/a)^\mu+(y/b)^\mu=1$, $a\geqq b>0$, be a Lamé curve with $\mu=2\nu(2\lambda+1)^{-1}$, where $\lambda\geqq 0$ and $\nu>0$ are integers. The rectangle parallel to the axes inscribed in C with largest area has vertices $D_i=(\pm a2^{-1/\mu}, \pm b2^{-1/\mu})$ and area $2^{2-2/\mu}$. For the points D_i the radius OD_i is the geometric mean of the segments F_jD_i, $j=1, 2$, where $F_j=\pm 2^{\frac{1}{2}-1/\mu}(a^2-b^2)^{\frac{1}{2}}$. For the Lamé hypersurface S: $\sum_{k=1}^n(x_k/a_k)^\mu=1$, $a_k>0$, where μ has the same form as above, the largest parallelepiped parallel to the axes inscribed in S has vertices with coordinates $x_k=\pm a_kn^{-1/\mu}$ and volume $2^nn^{-n/\mu}\pi a_k$. It is shown that the extremals of the isoperimetric problem for the Minkowski metric $ds=(dx^n+dy^n)^{1/n}$, where n is an integer not less than 2 have the form $(x-A)^{n/(n-1)}+(B-y)^{n/(n-1)}=c$. Sufficiency conditions are not discussed, but are contained in the reviewer's solution of the isoperimetric problem for general Minkowski metrics [see Amer. J. Math. **69**, 863–871 (1947); these Rev. **9**, 372]. *H. Busemann.*

Fenchel, W. A congruence theorem for convex polyhedra. Mat. Tidsskr. A. **1949**, 35–43 (1949). (Danish)

Historique et démonstration simple du résultat suivant, énoncé par Cauchy. Les polyhèdres convexes admettant un dévelopment (plan) donné sont égaux ou symétriques (s'ils existent). *J. Favard* (Paris).

Hadwiger, H. Über beschränkte additive Funtionale konvexer Polygone. Publ. Math. Debrecen 1, 104–108 (1949).

All functionals $\varphi(A)$ defined on all closed convex polygons A in E_2 and with the following 3 properties are determined: (1) $\varphi(A)=\varphi(A')$ if A' originates from A by a translation; (2) $\varphi(A+B)=\varphi(A)+\varphi(B)-\varphi(AB)$, where $A+B$ is a convex polygon decomposed by the chord AB into A and B (here AB is to be considered as the closed polygon obtained by traversing the segment AB in two directions); (3) $|\varphi(A)|$ is bounded for all A in the unit square. As an application a result of Blaschke [Vorlesungen über Integralgeometrie, vol. II, Teubner, Leipzig-Berlin, 1937, § 43] is derived which determines the same functionals with (1) replaced by $\varphi(A)=\varphi(A')$ for congruent A and A'. *H. Busemann.*

Obrešković, N. On hyperbolic integral geometry. C. R. Acad. Bulgare Sci. Math. Nat. 2, no. 2–3, 1–4 (1949). (Russian)

The author considers some elementary properties of integral geometry in the Lobačevskiĭ plane and the Lobačevskiĭ space. In both cases the model of Poincaré is used by which, in the two-dimensional case, properties of linear transformations in the upper part of the complex plane are studied. *H. A. Lauwerier* (Amsterdam).

Santaló, L. A. On parallel hypersurfaces in the elliptic and hyperbolic n-dimensional space. Proc. Amer. Math. Soc. 1, 325–330 (1950).

Herglotz [Abh. Math. Sem. Hansischen Univ. 15, 165–177 (1943); these Rev. 7, 475] has extended Steiner's formula for the length of a curve parallel to a given curve to an expression for the area of a hypersurface parallel to a given hypersurface in a space of constant curvature S_n. The present paper gives a new proof of this result and moreover obtains analogous expressions for the integrated mean curvatures of the parallel hypersurface in terms of the corresponding curvatures of the given hypersurface. Let $M_i(\lambda)$ be the integrated mean curvatures of a hypersurface at a constant distance λ from a hypersurface whose corresponding curvatures are M_i. Then $M_i(\lambda)=\sum_{k=0}^{n-1}M_k\varphi_{ik}(\lambda)$, where in the elliptic case

$$\varphi_{ik}(\lambda)=\sum_h\binom{n-1-k}{i-h}\binom{k}{h}\sin^{i+k-2h}\lambda\cos^{n-1-i-k+2h}\lambda,$$

summed over all meaningful values of h. In the hyperbolic case $\varphi_{ik}(\lambda)$ has a similar expression except that the circular functions are replaced by hyperbolic functions. Applications of this result are as follows. (1) In the elliptic case consider the polar hypersurface $(\lambda=\pi/2)$ whose curvatures are M_i^P; then $M_i^P=M_{n-1-i}$. (2) For a hypersurface of constant width Δ, $M_i=(-1)^{n-1-i}\sum_{k=0}^{n-1}M_k\varphi_{ik}(-\Delta)$. Numerous special cases are treated. *C. B. Allendoerfer* (Haverford, Pa.).

Algebraic Geometry

*Godeaux, Lucien. **Géométrie algébrique.** Sciences et Lettres, Liège. Tome 1, 236 pp., 1948; tome 2, 210 pp., 1949.

Each of the two volumes of this work is divided into two parts, dealing respectively with (I) birational transformations, (II) hyperspatial projective geometry, (III) geometry on an algebraic curve, and (IV) algebraic plane geometry. A third volume, on algebraic surfaces, appears to be in contemplation.

Part I contains a lot of theory, much of which one feels would be more at home in some of the other parts, or at least could be read more intelligently by a student who already knew a good deal of the matter in the other parts. It deals with Cremona transformations in the plane and in three dimensions, some of the familiar types of low order being analyzed in great detail, with special consideration of the fundamental elements, of which the arithmetical properties are also given with considerable generality. These are used to resolve singularities of algebraic plane curves, and surfaces in space; for the former, the Nöther-Chisini transformation into a curve with only multiple points with distinct tangents, and the Halphen transformation into one with only double points, are given, and for the latter, biplanar and uniplanar double points are classified (somewhat defectively as regards the uniplanar, since only the three types found by Cayley for cubic surfaces would appear to exist). The genus of a plane curve is also defined in terms of order and singularities, and it is shown that this cannot be negative, and that if it is zero the curve is rational.

Part II begins with a brief study of projective space of n dimensions (defined, of course, by means of complex homogeneous coordinates) and its subspaces, and of projective and dual transformations. Hypersurfaces and their elementary polar properties and dual treatment as envelopes come next, followed by varieties of lower dimensions which are treated by the Cayley-Halphen representation as the intersection of a cone and a number of monoids residual to a set of generators of the cone. Rational curves receive a chapter to themselves; those of order $\leqq 6$ are treated in detail and linear series on a rational curve studied (under the odd name of "involutions," whatever their dimensions). Rational surfaces also receive a chapter, containing the general theory of the plane mapping, and detailed study of the rational ruled surfaces and the Veronese and Steiner surfaces. A further chapter is given to Segre varieties, representing the sets of k points, each varying independently in a linear space. The V_4^6 representing the pairs of points of two planes is treated at length. The final chapter deals with the representation of three-dimensional line geometry on a quadric in five dimensions.

Part III (in the second volume) is devoted to the classical theory of curves; linear series, the canonical series, and the Riemann-Roch theorems are treated in an elementary way which should be intelligible to beginners; the canonical curves of the first few genera are described projectively (it is odd, however, that in dealing with the curve of genus 6 the author appears unaware that the most general type is the quadric section of a quintic del Pezzo surface, as he treats the plane sextic with four and the septimic with nine double points as essentially distinct cases). Correspondence theory is treated first by classical algebraic methods; then Riemann surfaces and Abelian integrals are introduced and discussed in a manner that is both simple and thoroughgoing, and applied to correspondence theory.

In part IV linear systems of plane curves are studied from the point of view of Cremona-invariance; the general theory of the Jacobian and adjoint systems, and of successive adjoints is given. The continuous groups of Cremona transformations are considered, and the cyclic and involutary transformations are classified. Finally, the plane involutions of order 2 are studied in detail. Much of this last part consists of the amplification, from a more sophisticated point of view, of topics already studied in part I.

P. Du Val (Athens, Ga.).

*Godeaux, Lucien. **Applications de la théorie des involutions cycliques appartenant à une surface algébrique.** Colloque de géométrie algébrique, Liège, 1949, pp. 177–195. Georges Thone, Liège; Masson et Cie., Paris, 1950. 200 Belgian francs; 1400 French francs.

*Libois, P. **La synthèse de la géométrie et de l'algèbre.** Colloque de géométrie algébrique, Liège, 1949, pp. 143–153. Georges Thone, Liège; Masson et Cie., Paris, 1950. 200 Belgian francs; 1400 French francs.

*Atchison, William Franklin. **Virtual Sets on an Algebraic Curve as Contrasted with Abelian Function Theory.** Abstract of a Thesis, University of Illinois, 1943. ii+8 pp.

A. Clebsch has proved a sequence of theorems concerning contact curves in a plane [Leçons sur la géométrie, v. 3, Gauthier-Villars, Paris, 1883, pp. 229–252] by using transcendental methods. In this work the same results are obtained by purely algebraic methods, and in fact by means of the de Jonquières formula and the use of virtual sets of order zero. The author seems not to be acquainted with a deeper paper by Castelnuovo [Ist. Lombardo Sci. Lett. Rend. (2) **25**, 1189–1205 (1892) = Memorie Scelte, pp. 79–94] where a similar algebraic method has been developed [cf. also F. Severi, Trattato di geometria algebrica, vol. I, part I, Zanichelli, Bologna, 1926, pp. 281–284].

B. Segre (Bologna).

Bagchi, Haridas. On the invariant relation between two cognate sextactic conics of a cubic. Proc. Indian Acad. Sci., Sect. A. **31**, 16–20 (1950).

Togliatti, Eugenio. Sulle superficie monoidi col massimo numero di punti doppi. Ann. Mat. Pura Appl. (4) **30**, 201–209 (1949).

It is known [Rohn, Math. Ann. **24**, 55–151 (1884)] that a monoid surface of order n can have at most $\frac{1}{2}n(n-1)$ double points besides its $(n-1)$-ple point. The present paper shows that for every value of $n \geqq 3$ the monoids with this maximum number of double points fall into two distinct families. One has dimension $\frac{1}{2}(n+2)(n+3)$, and there is nothing remarkable about the distribution of the double points. The other, for odd values of n, has dimension $\frac{1}{8}(n+3)(5n+1)+3$, the double points being all in the first neighborhood of the $(n-1)$-ple point, and for even values of n has dimension $\frac{1}{8}n(n+2)+6$, the double points being all on an $\frac{1}{2}n$-ic curve along which the surface is touched by a plane.

P. Du Val (Athens, Ga.).

Todd, J. A. On the quartic combinant of a pencil of quadric surfaces. Ann. Mat. Pura Appl. (4) **28**, 189–193 (1949).

The paper describes a geometric interpretation of a combinant of a pencil of quadric surfaces. If a pencil of quadrics, $\lambda_0 S+\lambda_1 S'=0$, is expressed symbolically as

$$a_x^2\alpha_\lambda = b_x^2\beta_\lambda = c_x^2\gamma_\lambda = \cdots,$$

the form $(acde)(bcde)(\alpha\gamma)(\beta\delta)a_xb_xe_\lambda$ is a covariant pencil of quadrics of the form $\lambda_0 Q+\lambda_1 Q'$. The combinant is the eliminant $SQ'-S'Q$ of the two pencils. It is known that for a g_4^1 on a twisted cubic Σ the tetrahedra whose vertices are the various sets of the g_4^1 are self polar with respect to a quadric K. There is another concomitant quadric K' obtained as follows. If P is a point on Σ and G a set of the g_4^1, then P has a polar plane with respect to the tetrahedron G. For fixed P and varying G, these polar planes form a pencil with axis $l(P)$. For fixed G and varying P, they form another pencil with axis $l(G)$. The lines $l(P)$ and $l(G)$ are the two sets of generators of a quadric K'. The author proves that if the g_4^1 is such that it contains the Hessian of one of its sets (and therefore the Hessian of each of its sets) then the quadrics K and K' coincide. The author then considers an 8:1 correspondence between the spaces $(x_0, \cdots, x_3)$ and $(y_0, \cdots, y_3)$, where $x_i^2 = y_i$ and the y_i are coordinates relative to a tetrahedron of reference G on the twisted cubic Σ. The pencil of planes $l(G)$ then corresponds to the pencil of quadrics $\lambda_0 S+\lambda_1 S'$. A pencil of planes $l(H)$ defined similarly with respect to the Hessian corresponds to $\lambda_0 Q+\lambda_1 Q'$. The quadric K then corresponds to the combinant quartic, $SQ'-S'Q=0$. He shows that the 8:1 correspondence arises naturally from a consideration of the 9-dimensional space which maps the quadric envelopes of the space $(x_0, \cdots, x_3)$.

D. E. Littlewood (Bangor).

Room, T. G. Self-transformations of determinantal quartic surfaces. I. Proc. London Math. Soc. (2) **51**, 348–361 (1950).

The topic of this set of four papers [see the three succeeding reviews] is a set of three determinantal quartics Π_x, Π_y, Π_z with equations

$$|a_{\alpha\beta\gamma}x_\alpha| = 0, \quad |a_{\alpha\beta\gamma}y_\beta| = 0, \quad |a_{\alpha\beta\gamma}z_\gamma| = 0,$$

where $(x_\alpha) = (x_0, x_1, x_2, x_3)$, (y_β), (z_γ) are homogeneous coordinates in three three-dimensional spaces. Between these are birational transformations $\omega_{xy}: a_{\alpha\beta\gamma}y_\beta z_\gamma = 0$, etc., whose resultant $\Omega_x = \omega_{xz}\omega_{zy}\omega_{yx}$ is a birational self-transformation of Π_x; ω_{xz} and $\omega_{zy} = \omega_{yz}^{-1}$ make the plane sections of Π_x and Π_y correspond on Π_z to two systems of Schur sextics of genus 3, which are residual with respect to cubic sections of the surface; and either transformation is induced on the surface by a Cremona transformation whose homaloids are cubics through a general curve of the system produced by the other [Schur, Math. Ann. **18**, 1–32 (1881); Jessop, Quartic Surfaces . . . , Cambridge University Press, 1916, chap. 9]. The sequences of systems produced from a given one by repetitions of Ω_x and its inverse, and the sequences on the three surfaces produced by cyclic repetitions of ω_{yx}, ω_{xy}, ω_{xz}, are studied in detail, and a systematic notation introduced (which is rather complicated, and in the reviewer's opinion not very clearly explained). Recurrence formulae are found for the orders of successive systems in a sequence and special study given to the sequences generated by the plane sections of each surface, and by the two systems of ∞^{11} curves of order 10 and genus 11 on each surface, which are residual with respect to quartic sections to the two systems of Schur sextics. In this first paper attention is confined to the general surface Π_x, on which the plane sections and either system of Schur sextics form a base.

P. Du Val (Athens, Ga.).

Room, T. G. Self-transformations of determinantal quartic surfaces. II. Proc. London Math. Soc. (2) **51**, 362–382 (1950).

The study of paper I [see the preceding review] is continued to the case in which Π_x contains a line. This may arise in four ways according as in the determinant $|a_{\alpha\beta\gamma}x_\alpha|$ the functions (i) throughout one row, (ii) common to two rows and three columns, (iii) common to three rows and two columns, or (iv) throughout one column, are linearly dependent on two only. To a line of type (iv) on Π_x corresponds a double point on Π_x and a line of type (i) on Π_y. The transform of the line on Π_x by $\Omega_x = \omega_{xy}\omega_{yz}\omega_{zx}$ is a twisted

cubic, not meeting the line. Similarly there is a cubic on Π_y, the transform by Ω_y^{-1} of the line on Π_y. There is an involution on Π_z traced by lines through the double point, and this corresponds to involutions on Π_y, Π_z, traced by chords of the cubics on these surfaces; by successive applications of the transformatives these give rise to endless sequences of involutions on all three surfaces. The orders of the isolated curves in the sequence generated by the line are studied, and the case where there are two skew lines of the same kind on Π_z (and hence two distinct nodes on Π_z) is considered. In this case there is a further type of involution on Π_z traced by common transversals of the two lines and this also gives rise to an endless sequence of involutions on all three surfaces. *P. Du Val* (Athens, Ga.).

Room, T. G. Self-transformations of determinantal quartic surfaces. III. Proc. London Math. Soc. (2) **51**, 383–387 (1950).

Continuing the topic of papers I, II [see the two preceding reviews], if Π_z has a line of type (ii), this corresponds to a conic on Π_y, the plane of which traces on Π_y a second conic, which corresponds to a rational quintic on Π_z of which the line is the unique quadrisecant. On Π_z the first conic corresponds to a rational quintic and the second to its quadrisecant line, which is of type (iii). These combinations quintic+quadrisecant line form degenerate Schur sextics. There are two other types of degenerate Schur sextic, occurring in pencils, on Π_y and Π_z, corresponding to the degenerate plane sections of Π_z consisting of the line and a cubic; these are on Π_y, a conic and an elliptic cubic, on Π_z, the line and an elliptic quintic, with of course three common points in each case. There is an involution on Π_z traced by common transversals of the line and the rational quintic, and that on Π_z corresponding to it is an involution of the same kind. *P. Du Val* (Athens, Ga.).

Room, T. G. Self-transformations of determinantal quartic surfaces. IV. Proc. London Math. Soc. (2) **51**, 388–400 (1950).

This fourth paper [see the three preceding reviews] is concerned with the case in which four elements of one row and three in another in the determinant $|a_{\alpha\beta\gamma}x_\alpha|$ are linearly dependent on any two of them, so that the line defined by the vanishing of these is simultaneously of types (i), (ii). In this case there is also a node lying in the line. The line and the node correspond on Π_y to two lines of types (i), (iv), and on Π_z to a node and a line which is simultaneously of types (iii), (iv). The two lines on Π_y correspond by Ω_y, Ω_y^{-1} respectively to two coplanar conics. The sequences of curves generated by these, and the corresponding involutions, are discussed at length. *P. Du Val* (Athens, Ga.).

Piazzolla Beloch, Margherita. Sul numero delle falde delle superficie algebriche. Ann. Mat. Pura Appl. (4) **29**, 121–123 (1949).

A real sheet of an algebraic surface is called odd or even according as a general line (by an obvious slip of the pen the author has written "plane") meets it in an odd or even number of points. An even sheet is of the first or second kind according as it contains any odd circuits or not. The following results are proved. A nonsingular surface of order $2n$ cannot have more than $\frac{1}{2}(2n-1)(2n-2)+1$ even sheets of the first kind. If it has this maximum number there are no other sheets of any kind. For all values of n there exist nonsingular surfaces of order $2n$ having this maximum number of even sheets of the first kind. The proof of the third statement appears to the reviewer to be unsound (this does not of course mean that the theorem is false; it seems likely enough). The surface is constructed in the form $f+\lambda g_1 \cdots g_n=0$, where $f=0$ is a cone whose general plane section has the required number of even circuits, $g_1=0, \cdots, g_n=0$ are ellipsoids not meeting the cone in real curves, and λ is a small constant. But if the sign of λ is such that the surface is interior to all sheets of the cone, the circuits obtained, though of the required number, will be of the second kind (like a convex quadric interior to, and everywhere close to, a cone) while if it is of the opposite sign, the sheets, though of the first kind, will coalesce near the vertex of the cone and be of less than the required number. *P. Du Val* (Athens, Ga.).

Zappa, Guido. Alla ricerca di nuovi significati topologici dei generi geometrico e aritmetico di una superficie algebrica. Ann. Mat. Pura Appl. (4) **30**, 123–146 (1949).

On the Riemann surface of a curve of genus p, the maximum number of independent 1-cycles whose mutual (and self-) intersection numbers are all zero is equal to p. From Hodge's results [J. London Math. Soc. **8**, 312–319 (1933); **12**, 58–63 (1937)] the author deduces that, on the Riemannian of an algebraic surface, the maximum number of independent 2-cycles which have zero mutual and self-intersections, as well as zero intersection with a "prime section," cannot exceed $\max(2p_g, R_2-2p_g-1)$, where p_g is the geometric genus and R_2 the second Betti number. It is conjectured that this number is always $2p_g$, whence $R_2 \leqq 4p_g+1$.

It is also shown that, if on the Riemann surface of a curve of genus p there are $h>p$ 1-cycles whose mutual intersection numbers are all zero, the Riemann surface will be divided into k parts where $h-k=p-1$. There is a 1-complex associated with this decomposition of the Riemann surface whose edges represent the h 1-cycles and whose vertices represent the k parts of the surface bounded by them. This complex is also associated with a degeneration of the curve of genus p into k lines with h points of connection. If now a surface F_n in [3] can degenerate into n planes with c lines of connection (common to only two planes and not part of the limit of the double curve of F) and τ triple points of connection (common to three planes meeting two by two in lines of connection), it is shown that on F there are τ 2-cycles, c 3-chains whose boundaries are made up of these 2-cycles, and that F is divided by the 3-chains into n parts. There is an associated 2-complex Φ whose faces, edges, and vertices correspond respectively to the τ 2-cycles, c 3-chains, and n parts of F. The τ 2-cycles on F have zero mutual and self-intersections and zero-intersection with a plane section of F, and $p_a+1=\tau-c+n$, which is the Euler-Poincaré characteristic of Φ. Since Φ is connected, $r_0=1$ (r_i being the ith Betti number mod 2 of Φ), whence $p_a=r_2-r_1$. In a subsequent paper it will be shown further that $r_2=p_g$ and $r_1=q$. In this paper the author establishes some weaker results connected with these.

In one respect the case of surfaces differs from that of curves. If one were to start with any set of 2-cycles with the intersection properties described above, one could always construct a complex Φ'. It is not, however, known what further conditions the 2-cycles might have to satisfy in order that the characteristic of Φ' should be equal to p_a+1. *D. B. Scott* (London).

***Comessatti, A. Problemi di realtà per le superficie e varietà algebriche.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 15–41, Rome, 1943.

This is an historical and bibliographical report on the state of knowledge on the properties of algebraic varieties over the real field, regarded as constructs on the similarly defined varieties over the complex field; e.g., the circuits of a real curve are regarded as cycles on the Riemann surface of the curve given by the same equation in complex coordinates. The locus of real points on a complex algebraic variety is clearly the locus of fixed points on the variety of an involutary self transformation of space, namely, the interchange of all pairs of conjugate complex points, and if the variety is definable by equations all of whose coefficients are real, this is a self transformation of the variety. In this case the variety is called real, even if it contains no real points. The necessary and sufficient condition that a variety be birationally transformable into a real variety is that it have an involutary self transformation which is antibirational, i.e., in which the coordinates of a point are rational functions of the complex conjugates of the coordinates of the corresponding points. If this condition is satisfied the variety is said to be of real type. The separation of real types within the same birational type thus depends on the classification of the involutary antibirational self transformations of the variety into classes whose members are birationally equivalent. E.g., the fact that there are two birationally distinct types of antihomographic transformations of a complex variable, reducible to $z' = \bar{z}$ and $z' = -1/\bar{z}$ (this last equation is misprinted without the minus sign) means that there are two distinct real types of rational curves, the former with one cycle of real points, the latter without real points. A variety with no birational self transformations can have at most one antibirational and hence can give rise to at most one real type.

The problems considered, mainly for algebraic surfaces, are, for each real type, the number and parity and topological properties of the real sheets. These latter are only relative invariants, being changed by a real birational transformation with real base points, and they are associated with the relative invariants of the surface in the complex field, such as the Zeuthen-Segre invariant. A summary is given (too condensed to be very clear without reference to the original papers) of the considerable body of results obtained by the author for rational surfaces, and by Lefschetz and the author for Abelian varieties, these two classes having been selected as having an infinity of birational self transformations. The paper ends with a review of the outstanding questions and the methods available and of several relations between the topology of the real sheets and that of the Riemannian fourfold of the surface. *P. Du Val.*

Hodge, W. V. D. Some recent developments in the theory of algebraic varieties. J. London Math. Soc. 25, 143–157 (1950).

Presidential address to the London Mathematical Society, 17 November 1949.

van der Waerden, B. L. Le théorème de Bézout pour les hypersurfaces. Ann. Mat. Pura Appl. (4) 30, 73–74 (1949).

Let $S_1, \cdots, S_n$ be n hypersurfaces in the projective n-space. Assuming that these hypersurfaces have only a finite number of common points, let d be the sum of the multiplicities of these points. Taking for granted results to the effect that d does not depend on which hypersurfaces of given degrees one considers, it is proved that d is the product of these degrees. *C. Chevalley* (New York, N. Y.).

***Severi, Francesco. Grundlagen der abzählenden Geometrie. Ins Deutsche übertragen von Wolfgang Gröbner.** Mathematische Forschung, Band 1, Heft 2. Wolfenbütteler Verlagsanstalt, Wolfenbüttel-Hannover, 1948. 125 pp.

This is a translation of the paper "I fondamenti delle geometria numerativa" [Ann. Mat. Pura Appl. (4) 19, 153–242 (1940); these Rev. 7, 476]. *C. Chevalley.*

Differential Geometry

***Vygodskil, M. Ya. Differencial'naya geometriya. [Differential Geometry].** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 511 pp.

Largely using vector and Gaussian notations, the author presents the classical theory of curves and surfaces in Euclidean (x, y, z)-space. The development follows the classical pattern, starting with elementary properties of space curves and advancing to a consideration of the moving trihedral, Frenet formulas, and intrinsic equations; then turning to a consideration of the theory of surfaces and the geometric properties associated with the first and second fundamental quadratic forms; and ending with a presentation of the fundamental formulas in the theory of surfaces and a discussion of intrinsic geometry. The book is distinguished by its excellent Russian typography and its wealth of examples, figures, and exercises.

E. F. Beckenbach (Los Angeles, Calif.).

***Struik, Dirk J. Lectures on Classical Differential Geometry.** Addison-Wesley Press, Inc., Cambridge, Mass., 1950. viii+221 pp. $6.00.

Cet ouvrage comprend cinq chapitres. Le 1er chapitre traite de la théorie des courbes gauches, dont il expose les notions fondamentales, sans négliger ni le point de vue imaginaire (plans, cônes, et courbes isotropes) ni le point de vue global (ovales et principaux théorèmes de géométrie intégrale qui s'y rapportent). Le 2ème chapitre aborde la théorie générale des surfaces par l'étude des deux 1ères formes fondamentales et des notions qui s'y rattachent (métrique, courbure, réseaux conjugués, systèmes triples orthogonaux, coordonnées elliptiques, etc.). L'auteur ne manque pas, lorsqu'il le juge à propos, d'indiquer la véritable origine géométrique des notions introduites. Ainsi, à propos des lignes de courbure, il indique que Monge [Application de l'analyse à la Géométrie, 5ème ed., Bachelier, Paris, 1850] les a introduites dans un mémoire célèbre relatif au problème des déblais et remblais, problème dont il rappelle l'énoncé précis. Les notations, comme d'ailleurs dans le 1er chapitre et dans toute la suite de l'ouvrage sont celles de l'analyse vectorielle, mais le passage aux notations et aux méthodes du calcul tensoriel est soigneusement préparé par une série de problèmes particulièrement bien choisis. Les équations fondamentales du 2ème ordre de la théorie des surfaces font l'objet du chapitre III, qui débute par une comparaison des points de vue de Monge et de Gauss. Les équations de Gauss-Weingarten et de Codazzi suivent immédiatement, présentées sous la forme la mieux

adapteé au passage aux méthodes tensorielles. L'auteur donne à leur sujet les notions indispensables sur les invariants de déformation d'une surface, et en fait diverses applications, telles par example la déformation continue des helicoïdes minima, ou la caractérisation de la sphère comme surface dont tous les points sont des ombilics ou comme surface à courbure constante positive fermée sans singularités. Ce même chapitre contient les premières notions sur les coordonnées curvilignes de l'espace, et se termine par le théorème fondamental relatif à l'existence d'une surface admettant deux formes fondamentales données. Le chapitre IV traite de la géométrie sur une surface, soit des propriétés ne mettant en jeu que le ds^2 et par suite invariantes par déformation de la surface. Le rôle essentiel est tenu par les géodésiques. La forme géodésique du ds^2 conduit aux surfaces à courbure totale constante et au théorème de Minding sur leur isométrie pour une valeur donnée de la courbure. La géométrie des surfaces à courbure constante fournit une première ouverture sur les géométries non-euclidiennes, dont l'auteur fait un historique très documenté, mettant notamment en évidence l'importance des contributions apportées au sujet par G. Saccheri. Une étude approfondie du théorème de Gauss-Bonnet et de ses incidentes topologiques termine le chapitre. Le chapitre V traite de sujets spéciaux destinés à préparer la lecture des grands traités ou des travaux originaux: enveloppes, représentations conformes, isométriques, géodésiques, etc. Le problème général de la déformation est présenté sous les deux formes que lui ont données Bour et Minding, et, en ce qui concerne le problème de la représentation géodésique, mention est faite du théorème de Beltrami sur la representation géodésique plane des surfaces à courbure totale constante et des couples de surfaces associées de Liouville ou de Lie. L'étude des surfaces minima, commencée au chapitre II, est reprise du point de vue variationnel, et prolongée à la faveur des résultats généraux obtenus sur les représentations isométriques ou conformes. Un paragraphe est consacré au rôle des imaginaires en théorie des surfaces, et l'ouvrage se termine par l'énoncé d'un certain nombre de questions.

P. Vincensini (Marseille).

***Shouten, I. A. [Schouten, J. A.], and Stroĭk, D. Dž. [Struik, D. J.]. Vvedenie v novye metody differencial'noĭ geometrii. II. Geometriya. [Introduction to Recent Methods of Differential Geometry. II. Geometry.]** Gosudarstv. Izdat. Inostr. Lit., Moscow, 1948. 348 pp.

This is a Russian translation of vol. 2 of "Einführung in die neueren Methoden der Differentialgeometrie" [Noordhoff, Groningen, 1938]. The bibliography is enriched with titles of Russian publications and brought up to 1948. A preface is added in which the translators refer to the contributions of the "first period of activity of the Soviet tensor school of B. F. Kagan," notably the work of Širokov, Vagner, Norden, and Raševskiĭ. *D. J. Struik.*

Mirguet, Jean. Sur une classe de surfaces à points multiples. C. R. Acad. Sci. Paris **230**, 1250–1252 (1950).

In a previous note [C. R. Acad. Sci. Paris **230**, 48–50 (1950); these Rev. **11**, 393] the author defined sets generalizing Bouligand's orthosurfaces. In the present note this notion is applied to the geometrical study of the envelope of a family $\mathfrak{F}$ of plane curves $f(x, y, \lambda)=0$, Σ denoting the graph of $f(x, y, z)=0$ in E_3. The characteristic points are the projections of points of Σ, where the parallel to the z-axis is a paratingent. The singularities of the envelope are investigated in connection with the singularities of Σ, which is of the type described in the above mentioned note. The assumptions concerning the family $\mathfrak{F}$ are not explicitly stated; the assertions are followed or preceded by short proofs. *C. Y. Pauc* (Cape Town).

Simonart, Fernand. Sur les configurations hexagonales. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **36**, 268–284 (1950).

Consider a hexagonal 3-web H in a plane and a hexagon $P_1 \cdots P_6P_1$ constructed in the usual way by means of H along an arbitrary point P_0. If our 3-web is not a hexagonal one then this construction leads to a heptagon $P_1 \cdots P_6P_1'$ with $P_1' \neq P_1$, the points P_0, P_1, P_1' being collinear. The author shows by a straightforward computation that $\overline{P_1P_1'}$ is of the third order at least with respect to $\overline{P_6P_1}$. Requiring a higher order, one obtains the Thomson differential equation whose integral $F(x, y)=C$ leads together with $x=$constant and $y=$constant to a hexagonal 3-web. Some special cases of hexagonal 3-webs are also considered.

V. Hlavatý (Bloomington, Ind.).

Teixidor, J. On the tangent developable of a twisted cubic. Gaceta Mat. (1) **2**, 73–77 (1950). (Spanish)

Seifert, L. Sur la surface engendrée par les cercles osculateurs d'une courbe gauche. Časopis Pěst. Mat. Fys. **75**, D49–D58 (1950). (Czech. French summary)

L'auteur étudie les systèmes de lignes de courbure de la surface considérée et les surfaces développables engendrées par les normales le long de ces courbes. Il étudie surtout la surface engendrée par les cercles osculateurs d'une courbe gauche dont la courbure est constante. Pour ce cas l'auteur donne de nombreux résultats dont nous citons un seul example. Les lignes de courbure de la surface engendrée par les cercles osculateurs d'une hélice située sur un cylindre de révolution se projettent sur la base comme des courbes parallèles aux épicycloïdes engendrées par un cercle de rayon $r=\frac{1}{2}\rho(1-\sin\omega)$ roulant sur un cercle de rayon ρ, le rayon du cylindre. *H. A. Lauwerier* (Amsterdam).

Wunderlich, Walter. Über die polykonischen Loxodromen. Ann. Mat. Pura Appl. (4) **29**, 177–186 (1949).

Let Φ be a surface obtained by revolving a Cartesian oval about its axis of symmetry. Let F_1, F_2, F_3 be the ordinary foci and let O be the singular focus of Φ. Let k be a curve on Φ and let P be a point on the axis of revolution of Φ. Let $\bar{k}$ be the curve into which k is transformed by the customary isometry of the cone Pk into the Euclidean plane. By studying $\bar{k}$ for the various positions of P on the axis of revolution of Φ the author proves a result of Cesàro [Rend. Accad. Sci. Fis. Mat. Napoli (3) **9**, 73–89 (1903)], namely: If k is a loxodrome on F_ik then it is also a loxodrome on F_jk and F_lk; moreover, k is a geodesic on Pk for exactly one choice of P (which choice depends on k). In addition, the author obtains many new results for such curves k; for instance, $\bar{k}$ is the involute of a circle when $P=O$ and k is itself a Darboux curve on Φ. Also a representation of k by means of an elliptic integral is given. *W. van der Kulk.*

Lorent, H. Sur une famille de transformations des courbes et des surfaces. II. Bull. Soc. Roy. Sci. Liége **18**, 462–471 (1949).

The author continues his study of certain types of transformation of curves and surfaces begun in an earlier paper [same vol., 300–311; these Rev. **11**, 394]. A few more cases

of the transforms of conics are considered. In space the transformation may be described as follows. Let $A(x', y', z')$ be a given point. Revolve about the x-axis the parabola with vertex at the point $(2x', 0, 0)$ and focus at $(x', 0, 0)$. The bisector of the angle between the line OA and the xy-plane intersects the paraboloid of revolution in the point $B(x, y, z)$. The transformation to be considered is the relation between A and B of the form $x=x(x', y', z')$, $y=y(x', y', z')$, $z=z(x', y', z')$. This transformation is varied by replacing the paraboloid of revolution by a hyperboloid of revolution with two sheets, the vertices of the hyperbola generating the hyperboloid being at the points $(x', 0, 0)$ and $(2x', 0, 0)$. Letting A generate a surface S', B generates a surface S, some of whose properties can be described in terms of those of S'. *V. G. Grove* (East Lansing, Mich.)

Bernštein, S. N. On the curvature of surfaces. Uspehi Matem. Nauk (N.S.) **5**, no. 4(38), 132–133 (1950). (Russian)

Translation of a paper which appeared in Math. Ann. **60**, 434–436 (1905).

Ulčar, Jože. Eine geometrische Deutung der mittleren Krümmung. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **2**, 267–270 (1949). (Macedonian. Russian and German summaries)

To obtain the mean curvature, the author integrates the canonical representation $z(r, \vartheta)=\frac{1}{2}r^2\{R_1^{-1}\cos^2\vartheta+R_2^{-1}\sin^2\vartheta\}$ with respect to ϑ. *W. Feller* (Princeton, N. J.).

Mishra, R. S. Curves whose geodesic torsion is extremum. Proc. Benares Math. Soc. (N.S.) **9**, 37–41 (1947).

L'auteur détermine par leur équation différentielle quadratique les courbes d'une surface, qu'il désigne par lignes de torsion géodésique, le long desquelles la torsion géodésique est extrémum. Ces courbes forment une double famille ∞^1, identifiable avec le double système de courbes mentionnées par R. Behari [The Differential Geometry of Ruled Surfaces, Lucknow University Studies, no. 18, 1946; ces Rev. 9, 378] d'équation $J[f, J(f, \varphi)]=0$, où f et φ sont les deux tenseurs quadratiques fondamentaux de la surface. De cette remarque on déduit aussitôt que l'équation des lignes de torsion géodésique d'une surface S peut prendre la forme géométrique $2k=k_m$, où k et k_m représentent respectivement la courbure totale et la courbure moyenne en un point quelconque de S. Ces lignes forment sur S le réseau orthogonal bissectant le réseau des lignes de courbure, et, sur les surfaces minima, coïncide avec le système des lignes asymptotiques. Un rapprochement est fait avec les courbes caractéristiques [Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces, Ginn, Boston, Mass., 1909, p. 131] de la surface S. *P. Vincensini* (Marseille).

Terracini, Alessandro. Su certe quaterne di direttrici di una rigata. Ann. Mat. Pura Appl. (4) **30**, 267–275 (1949).

Soit R une surface réglée quelconque non développable; C_y, C_z; C_z, C_b deux couples de directrices de R. L'auteur appelle associés ces deux couples de directrices si, y, z, a, b étant sur une même génératrice, et α, β désignant les plans tangents à R en a, b, les deux faisceaux de droites $y\alpha$, $z\beta$ de centres y, z et de plans respectifs α, β, appartiennent à un même complexe linéaire. Il forme la condition analytique pour que C_y, C_z; C_z, C_b constituent un couple de directrices associées, puis, interprétant géométriquement le résultat, montre que la condition nécessaire et suffisante d'association est que les tangentes t_y, t_z à C_y et C_z aux points y et z, et les conjuguées t_a', t_b' des tangentes t_a, t_b à C_a, C_b en a et b appartiennent à une même demi-quadrique. La représentation sur la quadrique de Klein, et des résultats antérieurs de l'auteur [Scritti Matematici offerti à Luigi Berzolari, Pavia, 1936, pp. 449–478], conduisent à la considération de l'ordre d'approximation σ [pair $\geqq 2$ ou infini] de l'association de deux couples de directrices d'une surface réglée. Cette introduction de l'ordre d'association permet une étude approfondie du cas où l'un des couples de directrices associées, ou les deux couples, sont formés d'asymptotiques du deuxième système de la surface réglée envisagée.
P. Vincensini (Marseille).

Kruppa, Erwin. Zur Differentialgeometrie der Strahlflächen und Raumkurven. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **157**, 143–176 (1950).

With every point S of the striction curve s of a ruled surface R in Euclidean three-space the author associates an orthogonal coordinate system with S as origin by choosing the straight line on R through S as the first axis and the normal to R at S as the second axis. For these coordinate systems the author obtains a set of Frenet formulas. In these formulas three invariants occur, κ, κ_1, σ, and R is determined up to motions if κ, κ_1, and σ are given as functions of the arclength on s, where σ is the "striction," i.e. the angle between the first axis and the tangent to s at S. If σ is identically zero, then R is the surface of the tangents to s, and κ and κ_1 are the curvature and the torsion of s. Thus it is possible from theorems about ruled surfaces to obtain theorems about curves. Using the preceding tools the author gives a detailed study of R, proving many known results and obtaining new ones. The author also studies ruled surfaces R which may be considered to be generalizations of Bertrand curves. *W. van der Kulk.*

Marussi, Antonio. Sulla struttura locale del geoide, e sui mezzi geometrici e meccanici atti a determinarla. Ann. Triestini. Sez. 2. (4) **2(18)**, 41–69 (1949).

It is pointed out in a systematic development that, for a potential field W, geometrical elements such as the radii of curvature of an equipotential surface, and mechanical elements such as the gravitational gradient, can be expressed locally as simple expressions in terms of the vector $g=\text{grad } W=dW/dp$ and the tensor d^2W/dp^2.
E. F. Beckenbach (Los Angeles, Calif.).

Marussi, Antonio. Fondements de géométrie différentielle absolue du champ potentiel terrestre. Bull. Géodésique N.S. **1949**, 411–439 (1949).

Continuing the work of the paper reviewed above, the author introduces a general system of space coordinates (longitude, latitude, and dynamical coordinate), and uses the methods of the absolute differential calculus to find, in their general form, the conditions of integrability which the local measurements must satisfy. *E. F. Beckenbach.*

Hristow, Wl. K. Über die Grössen M und N in der Geodäsie. C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 2–3, 61–64 (1949).

In higher geodesy, two important coordinate systems for an arbitrary surface are the system of geodesic coordinates, with $ds^2=n_P^2dp^2+dq^2$ and the system of geodesic polar coordinates, with $ds^2=d\sigma^2+m_P^2d\alpha^2$, the subscript P referring to the origin, or pole, of coordinates. Let now A and B be any two points of a surface S, joined by a geodesic line L,

and construct geodesic and geodesic polar coordinate systems relative to these points, with geodesic parallels normal to L. The author shows that we always have $m_A(B)=m_B(A)$, while in general we have $n_A(B)\neq n_B(A)$.

E. F. Beckenbach (Los Angeles, Calif.).

Voderberg, H. Neue Beiträge zur ersten Hauptaufgabe der Geodäsie für eine Drehfläche, insbesondere das Erdsphäroid. J. Reine Angew. Math. **187**, 153–168 (1950).

On a smooth surface of revolution, let there be given a point P_1 with "geographic" coordinates (B_1, L_1), and an arc C of length s issuing from P_1 in the direction with "azimuth" angle α_1. It is required to determine the coordinates (B_2, L_2) and angle α_2 for the terminal end-point P_2 of C. From a system of second order partial differential equations for B and L, the author obtains the solution as a set of three power series in s, deriving general expressions for the coefficients from recursion formulas. *E. F. Beckenbach.*

Backes, Fernand. Une généralisation des congruences de sphères cycliques. C. R. Acad. Sci. Paris **230**, 1569–1571 (1950).

[The author's name is misprinted Bakes in the original.] Demoulin called a congruence of spheres a cyclic congruence if the generic sphere possesses a triple infinity of circles generating the system. The purpose of this note is to exhibit a congruence of spheres possessing a quintuple of circles having focal spheres. *V. G. Grove.*

Jonas, Hans. Mit einem beliebigen Paare von Flächen gleichen konstanten Krümmungsmasses verknüpfte räumliche Beziehungen. J. Reine Angew. Math. **186**, 193–220 (1949).

Let $\mathfrak{r}(\alpha,\beta)$ and $\mathfrak{r}^*(\alpha,\beta)$ be two pseudospherical surfaces (with the same Gauss curvature -1) referred to the same asymptotic parameters α,β. Let $\mathfrak{N}$ and $\bar{\mathfrak{N}}$ be the normals to $\mathfrak{r}$ and $\mathfrak{r}^*$ respectively in corresponding points (e.g., in points with the same values of parameters α,β). The author finds first four points $\overset{i}{\mathfrak{r}}$ ($\overset{i}{\mathfrak{r}}^*$) on $\mathfrak{N}$ ($\bar{\mathfrak{N}}$) ($i=0,1,2,3$), such that the surfaces $\overset{i}{\mathfrak{r}}$ and $\overset{i}{\mathfrak{r}}^*$ are isometric, and, moreover, the surfaces $\overset{i}{\mathfrak{R}}=\frac{1}{2}(\overset{i}{\mathfrak{r}}+\overset{i}{\mathfrak{r}}^*)$ have α,β for asymptotic parameters. Furthermore, the lines $\overset{03}{P}=\overset{0}{\mathfrak{R}}\overset{3}{\mathfrak{R}}$, $\overset{12}{P}=\overset{1}{\mathfrak{R}}\overset{2}{\mathfrak{R}}$, going through $\frac{1}{2}(\mathfrak{r}+\mathfrak{r}^*)$ bisect the angle $\sphericalangle\mathfrak{N}\bar{\mathfrak{N}}$. They constitute two W congruences with the focal sheets $\overset{0}{\mathfrak{R}}$, $\overset{3}{\mathfrak{R}}$ and $\overset{1}{\mathfrak{R}}$, $\overset{2}{\mathfrak{R}}$ respectively. Denote by $\mathfrak{r}^{**}$ and $\mathfrak{r}^{*}$ the Bäcklund transforms of $\mathfrak{r}$ and $\mathfrak{r}^*$ (both Bäcklund transforms having the same characteristic constant). Then the corresponding W-congruences of lines $\overset{03}{P}{}^*$ and $\overset{12}{P}{}^*$ respectively are related to the W-congruences of $\overset{03}{P}$ and $\overset{12}{P}$ respectively by a W-transformation (e.g., the asymptotic lines of the corresponding focal sheets are related by a tangential congruence). These are the main results of the paper proved step by step by means of a straightforward computation. Related problems are also dealt with (for instance, the middle congruence of a four-Bäcklund cycle and so on). *V. Hlavatý.*

Cartan, Elie. Deux théorèmes de géométrie anallagmatique réelle à n dimensions. Ann. Mat. Pura Appl. (4) **28**, 1–12 (1949).

Ordinary real n-dimensional space can be closed by introducing one "point at infinity." Then we get the anallagmatic space. For this space $n+2$ supernumerary homogeneous coordinates can be used. Every real anallagmatic transformation can be written as a linear homogeneous transformation of the supernumerary coordinates leaving invariant a certain quadratic form. To every real hypersphere belong certain "semi-normal" coordinates, and the first theorem is concerned with the transformation of these special coordinates. To every pair of hyperspheres a set of anallagmatic invariants belongs. The second theorem gives the solution of the problem of determining $p\leqq n+1$ hyperspheres if the anallagmatic invariants of all pairs are given. As a final remark the difference between this investigation and similar investigations of W. Blaschke is made clear.

J. A. Schouten (Epe).

Sen, R. N. Corrections to my paper on 'Parallel displacement and scalar product of vectors. III.' Bull. Calcutta Math. Soc. **42**, 56 (1950).

Cf. the same Bull. **41**, 113–120 (1949); these Rev. **11**, 399.

Levine, Jack. Classification of collineations in projectively and affinely connected spaces of two dimensions. Ann. of Math. (2) **52**, 465–477 (1950).

In a space A_n with (symmetric) affine connection Γ^i_{jk}, a point transformation which changes each path of the space into a path and preserves (up to a linear transformation) affine parameters on paths is called an affine collineation (A.C.). In a space P_n with (symmetric) projective connection $\Pi^i_{jk}=\Gamma^i_{jk}-\{1/(n+1)\}(\delta^i_j\Gamma^a_{ak}+\delta^i_k\Gamma^a_{aj})$, a point transformation which changes each path of the space into a path is called a projective collineation (P.C.). If an A_n admits an r-parameter group G_r of A.C. with infinitesimal generators $X_af=\xi^i_af_{,i}$ (the comma denoting partial differentiation with respect to coordinates), then we have

$$(1)\quad X_a\Gamma^i_{jk}=\xi^i_{a,jk}+\xi^i_a\Gamma^i_{jk,l}-\xi^i_{a,l}\Gamma^l_{jk}+\xi^l_{a,j}\Gamma^i_{lk}+\xi^l_{a,k}\Gamma^i_{jl}=0.$$

If a P_n admits an r-parameter group G_r of P.C., then we have

$$(2)\quad X_a\Pi^i_{jk}=\xi^i_{a,jk}+\xi^i_a\Pi^i_{jk,l}-\xi^i_{a,l}\Pi^l_{jk}+\xi^l_{a,j}\Pi^i_{lk}+\xi^l_{a,k}\Pi^i_{jl}-\{1/(n+1)\}(\delta^i_j\xi^l_{a,lk}+\delta^i_k\xi^l_{a,lj})=0.$$

The purpose of this paper is to obtain all A_3's and all P_2's which admit real r-parameter continuous groups of collineations. The author uses Lie's classification of all real continuous groups in two variables into nonequivalent types and solves first equations (2) corresponding to each group for Π^i_{jk}. Then, to solve (1) corresponding to each group for Γ^i_{jk}, the author proceeds as follows. As (2) is written as

$$(3)\quad X_a\Pi^i_{jk}=X_a\Gamma^i_{jk}-\tfrac{1}{3}(\delta^i_jX_a\Gamma^l_{lk}+\delta^i_kX_a\Gamma^l_{lj}),$$

he solves first $X_a\Gamma^i_{lj}=0$ for Γ^i_{lj}, and then puts

$$\Gamma^i_{jk}=\Pi^i_{jk}+\tfrac{1}{3}(\delta^i_j\Gamma^l_{lk}+\delta^i_k\Gamma^l_{lj}).$$

The Γ's and Π's satisfying (1) and (2) respectively being obtained, to find all nonequivalent types of A_3 and P_2, he solves $X\Gamma^i_{jk}=0$ and $X\Pi^i_{jk}=0$ ($Xf=\xi^if_{,i}$) for ξ^i, the general solution being of the form $\xi^i=c^a\xi^i_a$, and obtains the complete groups of such collineations, that is, the groups of maximum number of parameters. Thus, he proves that there are 12 types of complete groups of A.C. and 4 types of complete groups of P.C. and he gives the list of these groups and of the corresponding components of connection.

He studies also groups of collineations in two-dimensional Riemannian spaces V_2. He shows that, if we exclude the space of constant curvature, the V_2 can admit only the

groups G_1, G_2, or G_3 of collineations and the corresponding fundamental quadratic forms of V_2. Finally he studies the groups of translatory collineations, that is, the groups of collineations whose trajectories are paths, and proves that if a V_2 admits one or more translatory collineations, it must be of constant curvature.

There are some misprints, of which we mention only the more serious ones: (2.7c) should be $y\Pi^1_{22}=0$; (2.7d) should be $3\Pi^2_{11}+y\Pi^2_{11}=0$; Π^1_{11} in the 5th line of p. 468 should be Π^2_{11}; and Γ^1_{12} in the second equation of (3.3) should be Γ^2_{12}.

K. Yano (Princeton, N. J.).

***Hlavatý, Václav. Contributo alla teoria delle connessioni.** Reale Accademia d'Italia, Fondazione Alessandro Volta, Atti dei Convegni, v. 9 (1939), pp. 479–506, Rome, 1943.

The theory of projective connexions can be treated in several ways. The author considers the methods introduced by (1) Cartan, based upon the notion of projective mapping of neighbouring local spaces, (2) T. Y. Thomas and Veblen, who studied the properties in A_n which depend only on the system of geodesics, and (3) van Dantzig, who uses homogeneous coordinates. These different theories are compared with each other and confronted with the work of Bortolotti and Hlavatý on the general theory of connexions [Ann. Mat. Pura Appl. (4) **15**, 1–45, 129–154 (1936)].

J. Haantjes (Leiden).

Cossu, Aldo. Su una notevole classe di varietà a connessione affine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 208–212 (1950).

Let $R^r{}_{\alpha\beta\lambda}$ be the curvature tensor of an A_n $(n>3)$, a^r, b^r, c^r three linearly independent vectors and (1) $v^r\equiv R_{\alpha\beta\lambda}{}^r a^\alpha b^\beta c^\lambda$. A necessary and sufficient condition for (2) $a^{[\alpha}b^\beta c^\lambda v^{\rho]}=0$ is $R^r{}_{\alpha\beta\lambda}=2\delta^r{}_{[\alpha}\varphi_{\beta]\lambda}+\delta^r{}_\lambda\psi_{\alpha\beta}$, where φ and ψ are functions of $R^\alpha{}_{\alpha\beta\lambda}$ and $R^\alpha{}_{\alpha\rho\sigma}$. (For a symmetric connection this condition is equivalent to A_n being projectively flat.) If the vector c^λ describes an infinitesimal cycle (by means of parallel displacement) in the bidirection of $a^{[\alpha}b^{\beta]}$ then according to (2) its final position is parallel to the three-direction of the three vector $a^{[\alpha}b^\beta c^{\lambda]}$. A similar statement holds for covariant vectors. The author discusses these statements in more detail by means of the tensors φ and ψ.

V. Hlavatý.

Hesselbach, Benno. Über die Erhaltungssätze der konformen Geometrie. Math. Nachr. **3**, 107–126 (1949).

By a variation principle a conservation law can be obtained from an invariant (Hilbert). This method is applied to invariants obtained from the conformal curvature tensor in a Riemannian space and leads to a tensor $U_{\lambda\lambda}$ for which $\nabla_\lambda U^\lambda{}_\mu=o(1)$. For $n=4$ the tensor $U_{\lambda\lambda}$ is conformal, invariant, and identical with the tensor found by Schouten and Haantjes [Math. Ann. **113**, 568–583 (1936), p. 576]. The author looks upon his equation (1) as the conservation law of electricity.

J. Haantjes (Leiden).

Vagner, V. V. On the theory of pseudogroups of transformations. Doklady Akad. Nauk SSSR. (N.S.) **72**, 453–456 (1950). (Russian)

A discussion of the foundations of the theory of pseudogroups in general and those of Lie in particular. The author applies these concepts to the question of differential geometric objects and proves that every Lie pseudogroup in a space X_n is the group of invariance of some differential geometric object in X_n.

M. S. Knebelman.

Martinelli, E. Sulla costruzione esplicita di forme differenziali con assegnati periodi. Ann. Mat. Pura Appl. (4) **30**, 97–121 (1949).

The differentiable manifold M_n under discussion is assumed to carry a Riemannian metric; and it is also assumed that dual bases $\Gamma^i{}_p$, $\Delta^i{}_{n-p}$ for the p- and $(n-p)$-cycles exist, which are made up of differentiable submanifolds of M_n. Consider a tube T around such a cycle Δ; if T could be represented as a product $\Delta\times C$, where C is a p-cell with coordinates $y_1,\cdots,y_p$, then a form of the type $\omega=f(s)dy_1\cdots dy_p$, where $s=(\sum y_i^2)^{\frac{1}{2}}$, where $f(s)$ vanishes on the boundary of T, and where ω is extended to all of M_n by setting it equal to 0 outside of T, would be closed, and the forms of this type, one for each basic cycle, would give a solution of the de Rham existence theorem. In general T is not such a product; the author uses Weyl's parallelism (along geodesics in Δ) to represent at least a portion of T around the point P as a product, and to obtain a form ω^P, similar to the ω above, valid in this portion of T. Then a form $\bar\omega$, the enveloping form of the ω^P, is defined such that at every point Q of T, it reduces to ω^P, where P is the foot of Q on Δ. This $\bar\omega$ can be expressed as a form in covariant differentials. With a suitable (explicit) choice of the factor $f(s)$ (cf. above), $\bar\omega$ is closed, and the problem solved. The forms of dimensions 1–4 are discussed in detail. Finally it is shown that the Wirtinger forms for complex projective space are obtainable in just this fashion.

H. Samelson.

Abramov, A. A. Topological invariants of Riemann spaces. Uspehi Matem. Nauk (N.S.) **5**, no. 2(36), 162–163 (1950). (Russian)

The author starts with a positive definite R_n with fundamental tensor $g_{\alpha\beta}$. The tensor field

$$\Omega_{i_1\cdots i_p}=F_{i_1\cdots i_p}\left(g_{\alpha\beta},\frac{\partial g_{\alpha\beta}}{\partial x^\gamma},\cdots,\frac{\partial^s g_{\alpha\beta}}{\partial x^\gamma\cdots},\cdots\right)$$

is assumed skew-symmetric and analytic in its arguments. Let C_p be a compact p-dimensional smooth, oriented subspace of R_n. Then $\Omega_{i_1\cdots i_p}$ is said to be a topological invariant if $\int_{C_p}\Omega_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p}$ is independent of $g_{\alpha\beta}$. Two tensor fields are equivalent if

$$\int_{C_p}\Omega_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p}=\int_{C_p}\bar\Omega_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p}.$$

If $\Pi_{i_1\cdots i_{2p}}=R^\alpha{}_{\beta\,[i_1i_2}R^\beta{}_{|\gamma|,\,i_3i_4}\cdots R^\lambda{}_{|\delta|,\,i_{2p-1}i_{2p}]}$ and for even n, $\bar\Pi_{i_1\cdots i_n}=g^{-\frac{1}{2}}R_{[i_1i_2,\,|\alpha_1\alpha_2|}\cdots R_{i_{n-1}i_n]\alpha_{n-1}\alpha_n}$, then the theorem is that tensor fields of the types $\sum C\Pi_{[i_1\cdots}\cdots\Pi_{\cdots i_p]}$ and $C\bar\Pi_{i_1\cdots i_p}$ are the only topological invariants up to an equivalence. For an affine space A_n let C_{p-1} bound a V_p and let there be given an n-repère E on C_{p-1}. For each E and $\Phi_{i_1\cdots i_p}$, there exists a skew-symmetric tensor field $\Psi_{\alpha_1\cdots\alpha_{p-1}}$ such that

$$\int_{V_p}\Phi_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p}-\int_{C_{p-1}}\Psi_{\alpha_1\cdots\alpha_{p-1}}du^{\alpha_1}\cdots du^{\alpha_{p-1}}$$

is independent of the affine connection and is independent of any continuous deformation of E.

M. S. Knebelman.

NUMERICAL AND GRAPHICAL METHODS

Hartley, H. O., and Pearson, E. S. Table of the probability integral of the t-distribution. Biometrika 37, 168–172 (1950).

A new table is constructed for Student's t distribution. This table gives the probability integral of t for ν degrees of freedom to five decimal places for $\nu=1(1)20$, $t=0.0(0.1)4.0(0.2)8.0$ and $\nu=20(1)24, 30, 40, 60, 120$, $t=0.00(0.05)2.0(0.1)4.0, 5.0$ ($\nu=a(b)c$ means that ν goes from a to c by steps of length b). For $\nu=1(1)10$ the 10^{-3}, 10^{-4}, 10^{-5}, $5\cdot10^{-6}$ percentage points of t are given.

H. Chernoff (Urbana, Ill.).

Bosse, Lothar. Tafel der Funktion

$$n=f(i,p)=\frac{90\,000\cdot(1-p)}{i^2\cdot p}$$

zur Bestimmung der Beobachtungszahl n, wenn die Zufallsfehlergrenzen $(3\cdot\sigma_x)$ einen bestimmten Prozentsatz i der vorgegebenen Wahrscheinlichkeit p nicht überschreiten sollen. Statist. Vierteljschr. 1, 38–39 (1948).

The table of the function mentioned in the title is for $p=.1(.1).9$ and for $i=.05$, $.1(.1)1(1)10(5)30(10)50$. Values of n are given to the nearest integer. No value is given when less than 100. *D. H. Lehmer* (Berkeley, Calif.).

***Lyusternik, L. A., Akušskii, I. Ya., and Ditkin, V. A. Tablicy Besselevyh funkcii. [Tables of Bessel Functions].** Mathematical Tables, no. 1. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 430 pp.

This volume contains seven tables. Table I: $J_0(x)$, $J_1(x)$, for $x=[0(.001)25; 7D]$, Δ^2. All of these values are implied in the more extensive Harvard tables [Tables of the Bessel Functions of the First Kind of Orders Zero and One . . . , Harvard University Press, 1947; these Rev. 8, 406]. Table II: Zeros α_k of $J_0(x)$, β_k of $J_1(x)$, γ_k of $J_1'(x)$, for $k=[1(1)40; 8D]$. The values for γ_k [7D for $k=1(1)10$] are mainly new. Tables III–VI: $J_0(\alpha_k x)$, $J_0(\beta_k x)$, $J_1(\beta_k x)$, $J_1(\gamma_k x)$, for $k=0(1)40$, $x=[.01(.01)1; 7D]$. These values are mostly new. Table VII: $2[J_1^2(\alpha_k)]^{-1}$, $2[J_0^2(\beta_k)]^{-1}=2[J_2^2(\beta_k)]^{-1}$, $2(1-\gamma_k^{-2})^{-1}[J_1^2(\gamma_k)]^{-1}$, for $k=[1(1)40; 7S]$. These new tables are for assisting in evaluating terms of the Fourier-Bessel expansions of a function. Tables III–VI and the table of $x\sin\pi x$ are also of use in this regard. The proofreading seems to have been done in a careless fashion; for example, page numbers are missing, equality signs are incomplete (p. 374), values of β_{40} on pp. 408–409 are grossly incorrect, the last entry of the second column of the table on p. 347 is wrong, and on p. 348, for

$$\frac{2}{J_2^2(\beta_k)}\int_0^1 xf(x)J_0(\beta_n x)dx$$

read

$$\frac{2}{J_1^2(\alpha_n)}\int_0^1 xf(x)J_0(\alpha_n x)dx.$$

R. C. Archibald (Providence, R. I.).

***Faddeeva, V. N., and Gavurin, M. K. Tablicy funkcii Besselya $J_n(x)$ celyh nomerov ot 0 do 120. [Tables of Bessel Functions $J_n(x)$ of Integral Orders 0 to 120].** Mathematical Tables, no. 2. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950. 439 pp.

In this volume are four tables. Table I gives values of $J_n(x)$, for $n=0(1)120$, $x=[0(.1)124.9; 7D]$, δ^2; these fill more than 370 pages of the volume. Apart from the modified second differences, the values through $n=78$, $x=99.99$ are already implied in published volumes of the Harvard tables [Tables of the Bessel Functions of the First Kind . . . , by the Staff of the Computation Laboratory, Annals of the Computation Laboratory of Harvard University, Harvard University Press, vols. 3–8, 1947; 9–11, 1948; 12–13, 1949; these Rev. 8, 406, 605; 9, 208, 307; 10, 150, 483; 11, 135, 463]. Table II gives 5D values less than 125, of zeros of $J_n(x)$. Thus there are 40 zeros for $J_0(x)$, 39 for $J_1(x)$, and so on to the last, a single zero of $J_{118}(x)$. Most of the values are new. Table III is devoted to coefficients in interpolation formulae. Table IV presents values of $J_n(x)$, $n=0(1)13$, $x=[0(.01)14.99; 8D]$; all these values are elsewhere available.

On page 4 the authors state that it was not until after their tables were completed that they saw the first 8 volumes of the Harvard Bessel function tables [loc. cit.]. When its final volume, soon to be published, has actually appeared, it will be found that in the Harvard series no zeros are listed, and values of the functions to at least 10D are given, and $n=0(1)135$, but x is never greater than 100, nor the interval less than unity for $n>85$. Thus these Russian tables contain new results. *R. C. Archibald.*

***Tables of Spherical Wave Functions.** Numerical Computation Bureau, Tokyo. Rep. no. 3. iv+77 pp. (1950).

Solutions of the wave equation considered involve Riccati-Bessel functions, tabulated and defined as follows:

$$F_l(x)=(\tfrac{1}{2}\pi x)^{\frac{1}{2}}J_{+(l+\frac{1}{2})}(x)=x^{l+1}f_l(x)/(2l+1)!!,$$
$$G_l(x)=(\tfrac{1}{2}\pi x)^{\frac{1}{2}}J_{-(l+\frac{1}{2})}(x)=(2l-1)!!g_l(x)/x^l,$$
$$F_l'(x)=-(l/x)F_l(x)+F_{l-1}(x)=(l+1)x^lf_l(x)/(2l+1)!!,$$
$$G_l'(x)=-(l/x)G_l(x)+G_{l-1}(x)=l(2l-1)!!g_l(x)/x^{l+1}.$$

The first large table is of F_l, G_l, F_l', G_l', for $l=0(1)6$, $x=[0(.05)10; 6D]$, δ^2. In limited ranges where F_l, F_l' have small significant numbers, and G_l, G_l' are too irregular to be interpolated, 6D values of ϕ_l, $\tilde{\phi}_l$, ψ_l, $\tilde{\psi}_l$ are also given, where

$$\phi_l=\log f_l(x)-\log(2l+1)!!,$$
$$\tilde{\phi}_l=\log f_l(x)+\log[(l+1)/(2l+1)!!],$$
$$\psi_l=\log g_l(x)+\log(2l-1)!!,$$
$$\tilde{\psi}_l=\log g_l(x)+\log[l(2l-1)!!].$$

For larger values, $l\geqq7$, recurrence formulae may be used. [The smallest earlier tabulated intervals were 0.1; see Math. Tables and Other Aids to Computation 1, 233 (1944).]

The next tables give 6D values of C_l, θ_l, $\tilde{C}_l$, $\tilde{\theta}_l$ or P_l, Q_l, R_l, S_l, where

$$G_l+iF_l=C_le^{i(x-\frac{1}{2}\nu\pi+\theta_l)},\quad F_l'-iG_l'=\tilde{C}_le^{i(x-\frac{1}{2}\nu\pi+\tilde{\theta}_l)},$$
$$F_l-iG_l=e^{i(x-\frac{1}{2}\nu\pi)}(-Q_l+iP_l),\quad G_l'+iF_l'=e^{i(x-\frac{1}{2}\nu\pi)}(R_l-iS_l),$$

$y=1/x$ in the range 0 to .1, at varying intervals .01, or .005, or .002, and δ^2, $l=1(1)6$. Finally, there are tables of f_l, $\tilde{f}_l$, g_l, $\tilde{g}_l$, for $l=1(1)6$, and $x=[0(.05)n; 6D]$, δ^2, n varying from 2 to 5.35. *R. C. Archibald* (Providence, R. I.).

Uhler, Horace Scudder. Miscellaneous hints for and experiences in computation. Scripta Math. 16, 31–42 (1950).

Kivikoski, E. Ein vernachlässigtes Interpolationsverfahren. Skand. Aktuarietidskr. 33, 39–87 (1950).

Verf. behandelt die sogenannte "verbesserte lineare Interpolation," die darin besteht, dass man als gesuchten Wert das arithmetische Mittel der durch die gewöhnliche lineare

Interpolation und durch die (lineare) Interpolation bei reziproken Argumenten gewonnenen Werte annimmt. Die Methode hat zwar nur beschränkte Anwendungsmöglichkeiten, hat aber den Vorteil, dass sich leicht eine obere Fehlergrenze bestimmen lässt. Die Arbeit verfolgt den Zweck, Kriterien für die Anwendbarkeit des Verfahrens aufzustellen und konkrete Fälle seiner Verwendung anzugeben. Insbesondere wird die Zweckmässigkeit der Methode bei finanzmathematischen Funktionen nachgewiesen.

P. Thullen (Panama).

Rutishauser, Heinz. Eine Konvergenzverbesserung für die Newtonsche Methode. Z. Angew. Math. Physik 1, 211–212 (1950).

L'auteur considère les équations en x: $f(x)=a$, $f(x)$ polynome fixe, a prenant un grand nombre de valeurs dans l'intervalle: $[f(x_0)-\Delta_0, f(x_0)+\Delta_0]$, Δ_0 petit. On applique plusieurs fois de suite à cette équation la méthode de Newton en partant de la valeur x_0. On s'interesse au nombre maximum d'opérations nécessaires pour obtenir une approximation donnée. L'auteur indique que la formule $x_{n+1}=x_n+[1/f'(x_n)][a-f(x_n)-\Delta_{n+1}]$ avec

$$\Delta_{n+1}=\tfrac{1}{2}\Delta_n^2 f''(x_0)/f'^2(x_0),$$

qui diffère de celle de Newton par le terme supplémentaire, en donne un nombre maximum d'opérations moins élevé que la méthode de Newton. Le travail supplémentaire dû au calcul des Δ est négligeable puisque ces quantités sont indépendantes de a. Cette méthode est utile pour résoudre les équations indiquées avec un programme de calcul fixe.

J. Kuntzmann (Grenoble).

Waugh, Frederick V. Inversion of the Leontief matrix by power series. Econometrica 18, 142–154 (1950).

The author approximates the matrix $(I-A)^{-1}$ by the sum of the first m terms of the power series $I+A+A^2+A^3+\cdots$ and determines the bounds of the error in the approximation. *W. E. Milne* (Los Angeles, Calif.).

***Hartree, Douglas R. Calculating Instruments and Machines.** The University of Illinois Press, Urbana, Ill., 1949. ix+138 pp. $4.50.

In 1948 the author delivered a series of lectures at the University of Illinois on various types of computing machines and on their use in solving mathematical and physical problems. He has with minor modifications now published these lectures in this book. There are three major topics treated in the text: analog machines; the first large automatic digital machines; and a discussion of the new machines now being developed and constructed. Under the first topic the author dwells principally on the differential analyser and gives not only a clear picture of the essentials of this instrument but also a detailed discussion of the solution of parabolic and hyperbolic partial differential equations on such an instrument. The discussion of digital machines is introduced by a chapter on the general principles underlying the design of fully automatic digital machines and then by a description of Charles Babbage's researches. The author then gives a brief description of the Harvard Mark I Calculator, the relay machines, the ENIAC, and the IBM Selective Sequence Calculator. He also has a section on the details of a calculation he carried out on the ENIAC. In the third part there is a discussion of the modern developments; and the book closes with a chapter on numerical analysis with particular attention to its relation to high-speed machines. The emphasis throughout the book is on the basic logical structure of machines and not on engineering details. It should prove a useful text not only to specialists in the field but also to those mathematicians who wish to gain an overall view of the field. There is also an excellent bibliography in the book. *H. H. Goldstine* (Princeton, N. J.).

Haupt, L. M. Solution of simultaneous equations through use of the A.C. network calculator. Rev. Sci. Instruments 21, 683–686 (1950).

Henrici, Peter. Zur Berechnung der Eigenwerte von Matrizen durch Lochkartenmaschinen. Z. Angew. Math. Physik 1, 185–189 (1950).

The author treats the problem of calculating the partial derivatives of the dominant eigenvalue of a matrix with respect to the elements of the matrix. The discussion is based on an iterative method of G. Kind-Schaad [Schweiz. Arch. Angew. Wiss. Tech. 13, 161–168 (1947); these Rev. 9, 211]. The author maintains that the derivatives can be calculated easily, without punching additional cards, by permuting the cards already needed for the iterative calculation of the eigenvalue itself. *H. B. Curry* (Louvain).

Richardson, L. F. A purification method for computing the latent columns of numerical matrices and some integrals of differential equations. Philos. Trans. Roy. Soc. London. Ser. A. 242, 439–491 (1950).

Soit K une matrice carrée d'ordre n à valeurs propres distinctes, I la matrice unité, X une matrice colonne quelconque. On peut décomposer X suivant les colonnes propres P_i de K. En multipliant X par le facteur $K-\lambda_i I$, λ_i valeur propre relative à P_i, on obtient une colonne dépourvue de composante suivant P_i. Si l'on multiplie X par le facteur $K-\rho I$, ρ voisin de λ_i, on fait disparaître la majeure partie de la composante suivant P_i. L'auteur étudie aussi l'effet sur X de la multiplication par $(K-\rho_1 I)(K-\rho_2 I)\cdots(K-\rho_p I)$. Supposant connus les ordres de grandeur de la plus grande et de la plus petite valeur propre, il donne pour $p=1, 2, 3$ les valeurs de $\rho_1, \cdots, \rho_p$ qui sont, en un certain sens, les meilleures pour une détermination précise de la colonne propre correspondant soit à la plus grande valeur propre, soit à la plus petite, soit à une valeur propre voisine d'un nombre s. La partie la plus délicate de la méthode est l'obtention de renseignements grossiers sur les colonnes et vecteurs propres. L'auteur montre sur des exemples comment s'y prendre. La méthode permet de traiter le cas de valeurs propres multiples, les diviseurs élémentaires étant ou non linéaires. L'idée directrice de cette méthode a déjà été utilisée par l'auteur pour la résolution approchée d'équations aux dérivées partielles [Philos. Trans. Roy. Soc. London. Ser. A. 210, 307–357 (1910)]. *J. Kuntzmann.*

Böning, P. Gerät zum Veranschaulichen der konformen Abbildung durch reziproke Radien. Kreisdiagramme der Wechselstromtechnik. Arch. Elektrotechnik 39, 388–390 (1949).

The inversion of a curve in the (X, Y)-plane with respect to the origin is accomplished graphically. The curve is traced by the end of a rod which is pivoted at a fixed point on the Z-axis. At this point of the rod a light is projected at right angles to the rod, the rod and the light beam lying in a plane normal to the (X, Y)-plane. The light beam will trace the inverse curve on the (X, Y)-plane. This device takes advantage of the fact that the altitude on the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse. *M. Goldberg.*

Hansen, Arthur G., and Yohner, Peggy L. A numerical procedure for designing cascade blades with prescribed velocity distribution in incompressible potential flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 2101, 51 pp. (1950).

The method outlined is for finding an aerofoil shape which, when placed in cascade with given inlet and outlet velocities and angles, shall have a specified distribution of velocity as a function of distance along its surface. First, in a conformal mapping of the flow region on the outside of a circle, the relation between surface velocity in the cascade plane and position in the circle plane is found. From this point onwards the method is similar to that of the reviewer [Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2104 (8766) (1945); these Rev. 8, 610], but considerable trouble is saved by avoiding the use of a certain untabulated definite integral. The core of the method is the combination of a conjugate harmonic function. This is done numerically, instead of analytically as in the reviewer's method. A procedure for this, based on 90 points evenly spaced around the unit circle, is found too inaccurate, and a procedure based on 360 points is therefore substituted. [Remark. It is wasteful to use a method in which the points are evenly spaced. Around the leading edge, where quantities change very rapidly, it is certainly necessary to use at least one point per degree, but away from this region only one point in every ten degrees is needed.] An example is worked out. No principles for guiding the choice of design specifications are given.

M. J. Lighthill (Manchester).

Winson, Jonathan. The solution of aero-elastic problems by electronic analogue computation. J. Aeronaut. Sci. 17, 385–395 (1950).

The details of the connections are given for the solution by a differential analyser of the REAC type (integration with respect to time only) of the differential equations of wing flutter. Since these are linear ordinary equations with constant coefficients, this is a straightforward, although complicated, application of the differential analyser. The author's treatment is not quite straightforward, because he starts with a partly integrated form of the flutter equations. Since the machine cannot take advantage of this analytical integration, but must go back to the original differential equation, the introduction of the partly integrated form appears to the reviewer to be an unnecessary complication, although it may have some advantages in the physical interpretation.

P. W. Ketchum (Urbana, Ill.).

Watanabe, Masaru, and Miita, Junichi. Application of the differential analyzer and its performance characteristics. Rep. Inst. Sci. Tech. Univ. Tokyo 3, 65–69 (1949). (Japanese)

Watanabe, Masaru, and Miita, Junichi. On the solution of Schrödinger equation obtained by a differential analyzer. Rep. Inst. Sci. Tech. Univ. Tokyo 3, 73–77 (1949). (Japanese)

Fröberg, Carl-Erik. On numerical computation of Coulomb wave functions. Ark. Fys. 2, 27–32 (1950).

L'auteur montre comment deux solutions de l'équation $u''+(1-(2\alpha/\rho)-l(l+1)/\rho^2)u=0$ écrites sous forme d'intégrales de Laplace peuvent être exprimées par des intégrales de variables réelles. Il indique que ces intégrales peuvent être utilisées pour le calcul numérique de valeurs de ces fonctions. Il signale par ailleurs que National Bureau of Standards Computation Laboratory a établi des tables (pas encore publiées) de ces fonctions. *J. Kuntzmann.*

Harkeevič, Yu. F. The graphical solution of partial differential equations of parabolic type. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 303–310 (1950). (Russian)

Fourier's heat equation $\partial u/\partial t=a^2\nabla^2 u$ with initial conditions and boundary conditions is reduced in the usual manner to a partial difference equation for values at the nodes of a rectangular screen. This difference equation is solved by an obvious step-by-step procedure. Three worked examples are given, one each for the one-, two- and three-dimensional problems. Actually, the work is done numerically rather than graphically. *W. E. Milne.*

Jaeger, J. C. A Schmidt mechanism for approximate solution of the equation of linear flow of heat in a medium whose thermal properties depend on the temperature. J. Sci. Instruments 27, 226–227 (1950).

Bell, D. A. Reactive circuits as computers and analogues. Electronic Engrg. 22, 232–235 (1950).

Osida, Isao. High precision nomogram. J. Phys. Soc. Japan 3, 25–30 (1948).

The author has rediscovered the alignment chart for $f_1+f_2=f_3$ which was explained and illustrated by L. R. Ford [Notre Dame Mathematical Lectures, no. 4, pp. 1–29, University of Notre Dame, Notre Dame, Ind., 1944; these Rev. 6, 53]. The method of construction is given for the case of a projective transformation of the equally spaced parallel line chart. A statistical study of relative errors found in actual use of the nomogram is reported.

R. Church (Annapolis, Md.).

Morita, Katuhiko. On the high precision alignment charts. J. Soc. Appl. Mech. Japan 2, 85–86, 110 (1949). (Japanese. English summary)

The method of the preceding paper is formulated in matrix terminology and extended to alignment charts in which no more than one of the three scales is curved. Assume the scales for z_1 and z_2 carried on the lines $x=\kappa$ and $x=\lambda$ respectively. Let $A_{\theta,\theta}$ denote the Massau matrix whose determinant (when set equal to zero) gives a chart of the desired accuracy. Suppose the desired ranges for the variables z_1 and z_2 correspond to $|y|\leqq(2n+1)\alpha$ where 2α is the height of the sheet available for plotting. Let $T_{r,s}$ denote the matrix of the affine transformation which carries the point $(\kappa, 0)$ into $(\kappa+r\tau, -r\alpha)$, (κ, α) into $(\kappa+r\tau, (1-r)\alpha)$ and $(\lambda, 0)$ into $(\lambda+s\tau, -s\alpha)$. Then, if the charts corresponding to the Massau matrices $A_{r,s}=A_{\theta,\theta}T_{r,s}$, $r, s=0, \pm2, \cdots, \pm2n$, are plotted for $|y|\leqq\alpha$ and superimposed, a single alignment diagram will be obtained including the desired ranges of z_1 and z_2. The procedure described is applied to the example [d'Ocagne, Traité de nomographie . . . , 1st ed., Gauthier-Villars, Paris, 1899, pp. 183 ff.] $z^2+pz+q=0$, $-24\leqq p\leqq24$, $-24\leqq q\leqq24$, and it gives a chart with p and q scales in three sections each, and the z scale in 8 sections (one of the $9=3\times3$ sections expected for the curved scale does not fall on the chart).

R. Church (Annapolis, Md.).

Krames, Josef. Über ein graphisches Verfahren zum gegenseitigen Einpassen von Luftaufnahmen. Österreich. Z. Vermessgswes. **37**, 13–29 (1949).

Part of the paper is a summary of the author's earlier work on the geometric properties of dangerous loci and dangerous regions connected with the problem of photogrammetric reconstruction [see Monatsh. Math. Phys. **50** 84–100 (1941); Monatsh. Math. **52**, 265–285 (1948); Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. **85**, 8–14, 25–31 (1948); these Rev. **6**, 15; **10**, 320; **11**, 126]. These properties are used to derive a graphical method for finding the mutual orientation of two aerial photographs.

E. Lukacs (Washington, D. C.).

Zwinggi, E. A study of the dependence of the premium on the rate of interest. Skand. Aktuarietidskr. **33**, 88–97 (1950).

The net premium for an endowment insurance is proportional to the quotient of two characteristic functions (Laplace transforms) with the force of interest as the argument, and its logarithmic derivative may thus be expressed approximately as a linear function of the interest with parameters involving the first two moments of the corresponding distributions. If interest is measured from a specified point the premium at one rate may be expressed in terms of that at another and of functions based on that rate.

H. L. Seal (New York, N. Y.).

MECHANICS

Martínez Salas, J. A point of view in the statics of material bodies without internal work. Revista Acad. Ci. Madrid **43**, 245–272 (1949). (Spanish)

This paper is a contribution to a new formulation of Newtonian mechanics in terms of modern mathematical concepts and without the logical difficulties which appear in the older expositions. In particular, the present author, and others working along the same lines, desire to avoid the procedure of initially formulating the fundamental concepts and laws for the case of a system consisting of a finite collection of particles, and then passing on to other cases by means of limiting processes which are not subjected to any very searching examination. The alternative procedure which they propose can be described briefly as follows.

A mechanical system is regarded as a point set $S(t)$, depending upon the time, in Euclidean 3-space. If t_1 and t_2 are two values of t, and if $A(t_1)$ is any subset of $S(t_1)$, there is a definite subset $A(t_2)$ of $S(t_2)$ corresponding to $A(t_1)$. Under these circumstances $A(t_2)$ is regarded as a transform of $A(t_1)$. On $S(t)$ there is defined a nonnegative completely additive set function with the property that the value of the function for any $A(t) \subset S(t)$ is independent of t. This set-function is regarded as representing a distribution of mass over the system; distributions of velocities, accelerations, forces, etc., are introduced similarly. The laws of mechanics are expressed as relations involving these set functions. By specializing the set $S(t)$ and other parts of the mathematical apparatus appropriately, we get the usual equations for the cases in which the system is a rigid body, a collection of particles, a membrane, a fluid, etc.

In the present paper the part of this program which relates to statics is developed rather extensively. Considerations of virtual work are employed to obtain the general conditions for equilibrium of a system; and the results appropriate to various particular classes of systems are derived by specialization, as indicated above. Mention should be made of the author's discussion of constraints, which is notably careful and explicit. *L. A. MacColl.*

***Koževnikov, S. N. Teoriya mehanizmov i mašin. [Theory of Mechanisms and Machines].** Gosudarstv. Naučno-Tehn. Izdat. Mašinostroit. Lit., Kiev-Moscow, 1949. 448 pp.

This advanced textbook, approved for use in Russian polytechnic institutes, offers a well rounded presentation of the modern condition of the title subject. Russian viewpoints are especially strong in the discussion of mechanism structure and classification. Mathematical developments are neither avoided nor favored, but graphical methods are not stressed. A list of chapter headings follows: structural analysis, four-bar linkages, a kinematic discussion of two-bar linkages, kinematics of simple open chains with dangling links, analytical kinematics of mechanisms, cam mechanisms, theory of toothed wheels, spatial gear transmission, gear mechanisms, simple spatial mechanisms with lower pairs, introduction to statics and dynamics of machinery, forces in machines, kinetostatics of mechanisms, friction in kinematic pairs, friction in higher kinematic pairs, friction of lubricated surfaces, friction transmission, transmission of work and power, efficiency, motion of mechanisms under external forces, nonuniformities of machine motion, regulation, balancing of rotating masses, balancing of inertia forces. *A. W. Wundheiler* (Chicago, Ill.).

Artobolevskiĭ, I. I. Progress in the Soviet theory of mechanisms and machines. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **4**, no. 16, 5–46 (1948). (Russian)

This survey covers Russian work beginning with a series of papers by Assur published between 1913 and 1918, and is divided into the following five sections: (1) structure (relations between the numbers of components and the freedom of the mechanism); (2) kinematics (relations between velocities, accelerations, etc.); (3) kinetostatics; (4) synthesis (design); (5) dynamics of machinery. "Synthesis" includes computing machinery and tolerance theory (the latter is claimed to be an exclusively Soviet development). Čebyšev's work is recalled with due insistence. The bibliography contains 209 items and is presumably complete until 1947. *A. W. Wundheiler* (Chicago, Ill.).

Levitskiĭ, N. I. On some transformations of plane mechanisms with lower pairs. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **2**, 17–33 (1947). (Russian)

The paper presents a method for deriving a new linkage L' whose one point traces the same curve as a known point of a given linkage L with turning or sliding pairs. It generalizes the substance of the Roberts theorem by using the Sylvester pantograph, i.e., the figure $M(OAO'A')M'$ where O is fixed, $OAO'A'$ is a parallelogram linkage,

$$\angle MAO' = \angle O'A'M' = \beta = \text{constant, and}$$
$$MA/AO' = O'A'/A'M' = 1/k = \text{constant.}$$

Then the M' curve is the M curve rotated through β and expanded k times. If OAO' (O fixed) is a two-bar component of any n-bar linkage L, and M a point of the plane AB, adjoin the parallel two-bar element $OA'O'$, consider in L any fully constrained sublinkage S containing OAO', turn

it about O through the angle β, expand it in the ratio k, merge M with M', and suppress the sublinkage S; this yields the transformed linkage L'. Examples and some considerations of the resulting changes in the dimensions are given. The author also discusses the number (or abundance) of the manifold of mechanisms generating the same curve, in terms of their link numbers. However, it is not clear to the reviewer whether the manifold is that of the linkages derived by means of the foregoing method, or of a more extensive manifold. *A. W. Wundheiler* (Chicago, Ill.).

Čerkudinov, S. A. On the general theory of geometric loci in metric synthesis. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 1, 181–216 (1947). (Russian)

The main achievement of the paper is a simple solution of the problem: Determine the four-bar linkages $ABCD$ (AD fixed) for which the ratio of angular velocities of AB and DC has given values at two positions, B_1C_1 and B_2C_2, given by the magnitudes of the angles $B_1AB_2=\beta$ and $C_1DC_2=\gamma$. The loci of all B_1 and C_1 are described in terms of "graphical elimination": each position of B_1 is determined as an intersection of two circles defined by A, D, β, γ and another parameter. The solutions of the corresponding problems for three and four given positions of BC are easily derived. This study completes (and apparently supersedes) an investigation started by H. Alt [Z. Angew. Math. Mech. 1, 373–398 (1921)].

The title applies only to the remaining 72 per cent of the paper, a collection of abstract remarks about geometric loci, couched in logically questionable symbolics. These remarks are exemplified in terms of "centers" of sets of corresponding points for n ($=2, 3, 4$) positions of a rigid plane. Loci of such centers are the basic tools of the author's technique in "metric synthesis" (=kinematic design).

A. W. Wundheiler (Chicago, Ill.).

Donato, Mancini. Prodotto di rotazioni finite attorno ad assi concorrenti. Atti Relaz. Accad. Pugliese Sci. N.S. 5, 121–126 (1947).

The author introduces some slight novelties into the familiar vectorial treatment of finite rotations in three-dimensional space, and uses the new formulae in a discussion of the resultant of two rotations about concurrent axes. *L. A. MacColl* (New York, N. Y.).

Sokolov, Yu. D. On infinite trajectories of a system of mass points under the influence of their mutual actions. Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. 1948, no. 10, 142–164 (1948). (Ukrainian. Russian summary)

The forces of mutual attraction ($f(r_{ij})<0$) or repulsion ($f(r_{ij})>0$) acting on three mass points are assumed to be equal to $m_im_j|f(r_{ij})|$, where $f(r)=dF(r)/dr$ is a real analytic function for real positive values of r. This function is finite and continuous for $r=0$ and satisfies the condition $\lim_{r\to+\infty} r^{1-2\alpha}f(r)=2\alpha$ [$\lim_{r\to+\infty} r^{-2\alpha}F(r)=1$, $\alpha>\frac{1}{2}$]. The author shows that, if the motion ceases to be regular at a certain moment $t=t_1$, $I^2=M^{-1}\sum m_im_jr^2_{ij}$ tends to a limit $I_1^2\geqq 0$, or to $+\infty$ when $t\to t_1$. He investigates the second case and gives the conditions corresponding to infinite trajectories. *W. Jardetzky* (New York, N. Y.).

Pogosov, G. S. Equations of motion for a system in non-holonomic-nonlinear constraints. Vestnik Moskov. Univ. 1948, no. 10, 93–97 (1948). (Russian)

Let Appell's equations, $\partial S/\partial\ddot{x}^i=X_i$, hold for a system, x, $\dot{x}$, and $\ddot{x}$ being kinematically independent (k. i.). If the reactions R_i of some new constraints satisfy (1) $a_\lambda{}^iR_i=0$, where $a_\lambda{}^i=\partial\ddot{x}^i/\partial\ddot{q}^\lambda$ with q, $\dot{q}$, $\ddot{q}$ being k. i., then, clearly, $\partial S/\partial\ddot{q}^\lambda=a_\lambda{}^iX_i=Q_\lambda$ (Q_λ are the generalized forces if the constraints are $\ddot{x}^i=a_\lambda{}^i\ddot{q}^\lambda+b^i$). The author proves this (rather laboriously) for constraints of the type $\ddot{x}=f(\dot{q}, q, t)$. However, the assumption (1) is strong enough to preserve Appell's equations even for constraints of the type $\ddot{x}=f(\ddot{q}, \dot{q}, q, t)$ with k. i. $\ddot{q}$, $\dot{q}$, q. *A. W. Wundheiler* (Chicago, Ill.).

Hydrodynamics, Aerodynamics, Acoustics

Polubarinova-Kočina, P. Ya. On sources and sinks on a surface. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 57–64 (1950). (Russian)

The present paper deals with the determination of the velocity potential and the stream function for the steady, incompressible, irrotational flow of a fluid on a surface due to a source (or sink) on the surface. The surface is supposed given in parametric form: $x=f_1(\alpha, \beta)$, $y=f_2(\alpha, \beta)$, $z=f_3(\alpha, \beta)$. If the parameters α and β are isothermal, that is

$$ds^2=\lambda(\alpha, \beta)[d\alpha^2+d\beta^2],$$

and if further $\lambda(\alpha, \beta)\equiv 1$, then the velocity potential φ and the stream function ψ of a flow on the surface due to a source at the point (α_1, β_1) both satisfy Laplace's equation $(\partial^2u/\partial\alpha^2)+(\partial^2u/\partial\beta^2)=0$, and hence

$$\varphi(\alpha, \beta)=\tfrac{1}{2}\log[(\alpha-\alpha_1)^2+(\beta-\beta_1)^2];$$
$$\psi=\arctan(\beta-\beta_1)/(\alpha-\alpha_1).$$

This remark is employed to determine φ and ψ for certain cylindrical surfaces and surfaces of revolution, and for a general ellipsoid. *J. B. Diaz* (College Park, Md.).

Friedman, Bernard. Theory of underwater explosion bubbles. Comm. Pure Appl. Math. 3, 177–199 (1950).

This work embodies and extends slightly the essential portions of a report which had only limited circulation by the author and M. Shiffman on the behavior of a pulsating gas bubble in the presence of a solid bottom and free surface. The theory rests on the assumptions that (1) the water is ideal and incompressible (thus, the effect of the initial shock wave after the underwater explosion is disregarded), (2) the gas bubble is spherical and expands adiabatically, (3) the water bottom is plane and rigid, (4) the free water surface is a plane equipotential (thereby neglecting the motion of the free surface). The bubble quantities of interest are in principle derived from integration of the Lagrangian equations for the bubble motion expressed in terms of the bubble radius and position, the total energy of the water bubble system being taken as the sum of the kinetic energy of the water, the potential energy of the displaced water and the internal energy of the bubble. To facilitate integration, the author makes the simplifying approximations (which have been checked in particular instances against numerical integration of the exact equations) that the bubble motion can be neglected during its expansion, and, more important, that the contribution of the bottom and surface to the kinetic energy of the water can be evaluated by means of only the kinetic energy of water motion due to a fixed expanding bubble. For the latter, the method of successive images gives a simple expression in terms of tabulated functions. These approximations permit the author to derive closed formulas for the

period of oscillation of the bubble, the distance its center moves during an oscillation, the maximum and minimum bubble radius, and finally the peak pressure emitted by the bubble. Calculated and experimental values for the period agree very well, while the predicted values for the peak pressure and bubble migration agree only moderately well with the experimental values. The author points out that there is considerable energy dissipated in the transition from shock wave stage to bubble motion and also between successive bubble oscillations, which cannot be explained by the theory. *D. Gilbarg* (Bloomington, Ind.).

Isaacson, E. Water waves over a sloping bottom. Comm. Pure Appl. Math. **3**, 11–31 (1950).

A progressive wave is supposed to travel along the surface of water bounded below by a plane sloping solid surface. A linearised boundary condition is applied at the free surface, and the wave energy is supposed to be totally absorbed by breaking in the region where the depth becomes smaller than the wavelength. This may be represented, although the theory becomes invalid as the amplitude increases in the said region, by a line sink of wave energy where the free surface meets the bottom. Solutions have been given previously when the angle between the bottom and the free surface is a rational multiple of π, and an asymptotic solution was derived by Friedrichs [Communications on Appl. Math. **1**, 109–134 (1948); these Rev. 10, 668] for the important case when this angle is small. In the present paper a solution is obtained in a closed form for arbitrary angle, and its connection established with the previous solutions for special angles. *M. J. Lighthill* (Manchester).

Sidrak, Sobhy. The drag on a circular cylinder in a stream of viscous liquid at small Reynolds numbers. Proc. Roy. Irish Acad. Sect. A. **53**, 17–30 (1950).

The author has improved Filon's solution [Proc. Roy. Soc. London. Ser. A. **113**, 7–27 (1926)] by Oseen's method of approximation for a viscous flow over a circular cylinder, by satisfying both the conditions at infinity and on the body. An expression for drag is derived, involving a set of constants which can be determined by solving a system of linear equations. A method of solution is given and the numerical values of drag coefficient for a range of Reynolds numbers are compared with previous results. *Y. H. Kuo.*

Sidrak, Sobhy. The flow of a viscous liquid past an elliptic cylinder. Proc. Roy. Irish Acad. Sect. A. **53**, 65–81 (1950).

The same technique [see the preceding review] is applied by the author to the case of an elliptic cylinder. The solution here is expressed in terms of Mathieu and modified Mathieu functions. The constants in the solution can similarly be determined by solving a system of linear equations. The numerical computation of drag coefficient has been carried up to a Reynolds number of 12 based on the major dimension. *Y. H. Kuo* (Ithaca, N. Y.).

Tsuji, Hiroshi. On the solution of the laminar boundary layer equations. Rep. Inst. Sci. Tech. Univ. Tokyo **3**, 239–241 (1949). (Japanese. English summary)

It is found that the boundary-layer equations can be reduced to ordinary differential equations if, and only if, $x(dM_1/dx)$ is expressed as a polynomial of the parameter $M_1=(x/U)(dU/dx)$, or M_1 as that of x. Several examples of U appropriate to this condition are presented. In particular, in the case $U=U_0x\exp(-x)$, namely Falkner's flow, the transformation of the equation is developed.

From the author's summary.

Betz, Albert. Reihendarstellung der Geschwindigkeitsverteilung in laminaren Grenzschichten. Arch. Math. **2**, 220–222 (1950).

The author suggests the use of the function

$$1-\exp\{-(b_1\eta+b_2\eta^2+b_3\eta^3+\cdots)\}$$

to represent the velocity distribution in a laminar boundary layer, where $\eta=yU/(2\nu\int_0^x U dx)^{\frac{1}{2}}$, x and y are the distances along and perpendicular to the solid boundary, U is the free-stream velocity, and ν is the kinematic viscosity coefficient. The coefficients $b_1, b_2, b_3, \cdots$ are to be determined by the conditions for $\partial^n u/\partial y^n$ at $y=0$. By comparing with the Hartree profiles, the author concludes that three or four terms of the series give the desired accuracy.

C. C. Lin (Cambridge, Mass.).

Hatanaka, Hiroshi. Theory of the laminar boundary layer on the surface of a body having an axial symmetry. Rep. Inst. Sci. Tech. Univ. Tokyo **3**, 115–117 (1949). (Japanese)

Watson, E. J. Asymptotic solution of a boundary layer suction problem. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2298 (9838)**, 8 pp. (1950).

Laminar boundary-layer flow over a flat plate is characterized by the equation $f'''+ff''=0$, where the velocity profile $u(y)$ is proportional to $f'(\eta)$, $\eta=\text{constant}\times yx^{-\frac{1}{2}}$, and $f(0)$ denotes the value $v_0(x)$ of the normal velocity at the plate. This was solved by Blasius for the impermeable plate, $f(0)=0$, and recently by several authors for porous plates with suction and blowing, taking $f(0)=k$ for several different values of k. The present author writes $f(\eta)=k+k^{-1}\phi(\zeta)$; $\zeta=\text{constant}\times v_0 y$ and obtains, successively, several terms of the asymptotic expansion appropriate for large k: $\phi=\phi_0+\phi_1k^{-2}+\phi_2k^{-4}+\cdots$. For example, the first term of this expansion gives the asymptotic boundary-layer profile $u=\text{constant}\times[1-\exp(-\zeta)]$. The calculation is carried to ϕ_2, and tabular values are given. The corresponding expressions for displacement and momentum thicknesses and shear stress are also worked out. The velocity distributions are compared, for several values of k, with Thwaites's [same Rep. and Memoranda no. **2241 (9656)** (1946); these Rev. 10, 76], which were obtained by using a differential analyzer.

W. R. Sears (Ithaca, N. Y.).

Prandtl, L., und Vandrey, Fr. Fliessgesetze normal-zäher Stoffe im Rohr. Ein Beitrag zur Rheologie. Z. Angew. Math. Mech. **30**, 169–174 (1950). (German. English, French, and Russian summaries)

The term "normal-zäh" introduced in this paper refers to fluids for which the shear rate γ and the shearing stress τ are related by means of $\gamma=C\sinh(\tau/A)$, where A and C are constants. The flow of such a fluid through a circular tube is discussed, and a method is indicated by which the constants A and C can be determined from observations of the volume rates of flow for two different pressure gradients.

W. Prager (Providence, R. I.).

Kampé de Fériet, J. Spectral tensor of homogeneous turbulence. Symposium on turbulence, 1 July 1949. Naval Ordnance Laboratory, White Oak, Md., Rep. **NOLR-1136**, pp. 1–31 (1950).

The author considers the four kinds of averages usually used in the study of homogeneous turbulence: (1) the statistical average; (2) the space average; (3) the time average; and (4) the space-time average. While the first two kinds are obviously the proper ones to use for theoretical investigations, the third kind is the one usually measured experimentally. In this theoretical investigation, the author shows that the spectral tensors defined by (1) and (2) cannot be assumed to be equal without introducing the unfortunate restriction that the turbulent quantities involved cannot be represented by Fourier-Stieltjes transforms, specifically almost periodic functions. The proof is essentially based on the fact that the spectrum (2) has jumps while the spectrum (1) is usually assumed to have a well-defined density.

C. C. Lin (Cambridge, Mass.).

Hama, Ryosuke. The energy distribution in the spectrum of turbulence. Rep. Inst. Sci. Tech. Univ. Tokyo **3**, 223–228 (1949). (Japanese. English summary)

The author calculates the spectrum of turbulence based on the following hypotheses. "(1) The energy of turbulence u_λ^2 of the particles of wave-length λ takes only discontinuous values, (2) the number of particles of wave-length λ, having an energy, is given by Maxwell-Boltzmann law." The exact physical meaning of these hypotheses is hard to understand. Furthermore, in comparing his results with experiments, the author does not seem to have realized that the one-dimensional spectrum usually measured does not represent the energy distribution (as was first clarified by Heisenberg).

C. C. Lin (Cambridge, Mass.).

Frenkiel, F. N. On turbulent diffusion. Symposium on turbulence, 1 July 1949. Naval Ordnance Laboratory, White Oak, Md., Rep. **NOLR-1136**, pp. 67–86 (1950).

The three-dimensional turbulent diffusion of particles suspended in a flowing fluid is studied by assuming the distribution of these particles to follow the error law when they are emitted from a source in a stationary fluid, the variance of the error-law distribution varying with time. Experiments are described for comparison with the theory.

C. C. Lin (Cambridge, Mass.).

Kiebel, I. A. Some studies on the flow of a gas in the region of transition through the velocity of sound. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1252**, 12 pp. (1950).

[Translated from Izvestiya Akad. Nauk SSSR Otd. Tehn. Nauk. **1947**, 253–259.] The partial differential equation for the stream function ψ in irrotational flow of a polytropic gas, after linearization by introducing the hodograph plane as the plane of independent variables, is reduced, following Christianovitch [Trudy Central. Aero-gidrodinam. Inst. no. **481** (1940); these Rev. **7**, 39], to the normal form $\psi_{ss}+\psi_{\bar{s}\bar{s}}+(d/ds)\ln K^1\psi_{\bar{s}}=0$ in the subsonic case, and to the normal form $\psi_{ss}-\psi_{\bar{s}\bar{s}}+(d/ds)\ln\chi^1\psi_{\bar{s}}=0$ for supersonic flow, and the graphs of the functions $K^1=K^1(s)$, $\chi^1=\chi^1(\bar{s})$ are given.

M. H. Martin (College Park, Md.).

Tomotika, S., and Tamada, K. Studies on two-dimensional transonic flows of a compressible fluid. II. Quart. Appl. Math. **8**, 127–136 (1950).

In this paper the authors give an alternative form of solutions to the simplified equations in the hodograph plane [see the authors, same Quart. **7**, 381–397 (1950); these Rev. **11**, 275]. It is shown that the solutions are expressible in terms of hypergeometric functions involving an arbitrary parameter. For special values of this parameter, the solutions reduce to elementary functions. The possibility of obtaining useful flow patterns in the physical plane by these solutions is indicated but no numerical example is presented.

Y. H. Kuo (Ithaca, N. Y.).

Levey, H. C. High-speed flow of a gas past an approximately elliptic cylinder. Proc. Cambridge Philos. Soc. **46**, 479–491 (1950).

This is an application of Cherry's method of transformation of plane steady potential flows of a compressible fluid [Proc. Roy. Soc. London. Ser. A. **192**, 45–79 (1947); these Rev. **9**, 544] to a specific flow past an approximately elliptic cylinder with arbitrary eccentricity and arbitrary angle of attack. The main contribution here is the mapping of the potential flow of an incompressible fluid about an elliptic cylinder between the hodograph and the physical planes. The mapping function has two branch-points, I and J, both of which correspond to infinity of the physical plane. By following Cherry's procedure, the solution for the compressible flow valid about the origin is continued into the region beyond the circle of convergence through I, say. In the case of a thin ellipse and large angle of attack, the solution is further continued into a region beyond the circle through J. Whether or not this step is required is not justified. No numerical results are included.

Y. H. Kuo.

Bers, Lipman. Velocity distribution on wing sections of arbitrary shape in compressible potential flow. III. Circulatory flows obeying the simplified density-speed relation. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2056**, 36 pp. (1950).

Assuming Chaplygin's simplified density-speed relation, which is equivalent to a linear approximation to the pressure-volume curve at stagnation conditions, the author is able to construct solutions to the potential equation of compressible flow which represent flow about airfoil profiles. This is an extension of previous work [same Tech. Notes no. **1006** (1946); these Rev. **8**, 107]. The shape of the profile, stream speed, and angle of attack may be specified. The problem is reduced to that of a nonconformal mapping of the region outside the airfoil into the region outside a circle. This mapping is accomplished by means of an integral equation somewhat analogous to the well-known one of Theodorsen and Garrick. A numerical procedure for approximate solution is worked out and a numerical example is given. Certain corrections, typographical and numerical, in the earlier paper [loc. cit.] are made.

W. R. Sears.

Miles, John W. A note on subsonic aerofoil theory. Aeronaut. Quart. **2**, 23–33 (1950).

"A linearized theory for the subsonic lifting surface is formulated in terms of Fourier integral solutions of Laplace's equation. The symmetric and antisymmetric problems of the first kind are solved explicitly, while the problems of the second kind depend on the solution of dual integral equations. The antisymmetric problem of the second kind is cast in a variational form, from which certain well-known theorems may be reduced." From the author's summary.

A. E. Heins (Pittsburgh, Pa.).

Miles, John W., and Naiman, Irven. Aerodynamic derivatives for an oscillating rectangular airfoil at supersonic speeds. Naval Ordnance Test Station, Inyokern, Calif. Tech. Memo. **RRB-32**, i+39 pp. (1949).

Sauer, Robert. Elementare Theorie des langsam schwingenden Überschallflügels. Z. Angew. Math. Physik **1**, 248–253 (1950).

A general solution to the equation of linearized, potential, unsteady supersonic flow past a thin, two-dimensional airfoil is obtained by neglecting powers of the frequency higher than the first. The result is in agreement with a rather different form obtained by the reviewer [J. Aeronaut. Sci. **16**, 378–379 (1949); these Rev. **10**, 755]. *J. W. Miles.*

Fenain, M. Trainée d'ailes delta symétriques à incidence nulle en régime supersonique. Recherche Aéronautique **1950**, no. 16, 27–38 (1950).

The fundamental problem considered is that of a thin wing whose surfaces are given by $x_3=\pm x_1^n f(|t|)$, where x_1, x_2, x_3 are rectangular Cartesian coordinates, x_1 is measured in the stream direction, and t denotes x_2/x_1. The function f is defined for $|t|\leqq\tau$, and $f(0)=1$, $f(\tau)=0$ is assumed. This was considered by Germain [Recherche Aéronautique **1949**, no. 7, 3–16 (1949); these Rev. **10**, 492]. By a suitable transformation the problem is put into a complex plane wherein certain nth derivatives of the velocity components appear as harmonic functions. Germain's results for this case are applied first to delta wings given by $x_3=\pm P(x_1)(1-t^p/\tau^p)$, where $P(x_1)$ is a polynomial such that $P(0)=P(1)=0$, and $p\geqq 1$. Numerical results, obtained in part by means of an electrolytic-tank analogy, are presented for several wings of this class (with $p=1$). These results include total drag and pressure distribution. The second class of wings considered comprises those that consist of two sweptback halves, each of these being a cone with vertex at the wing tip. These flows can be built up by superposition of elementary conical flows ($n=1$). The drag has been computed for the special case where the profile is polygonal; it agrees with the results obtained by Puckett and Stewart by another method [J. Aeronaut. Sci. **14**, 567–578 (1947); these Rev. **9**, 162]. *W. R. Sears.*

Kolodner, I. On the linearized theory of supersonic flows through axially symmetrical ducts. Comm. Pure Appl. Math. **3**, 133–152 (1950).

This is an account of axisymmetrical supersonic flow through ducts. The ground covered is much the same as in Ward's account [Quart. J. Mech. Appl. Math. **1**, 225–245 (1948); these Rev. **10**, 77], though rather more attention is given to the determination of special ducts for which the singularities in the linear theory do not arise. No attempt is made to elucidate the phenomena in the real flow to which the singularities, when they do occur, correspond.

M. J. Lighthill (Manchester).

Pack, D. C. A note on Prandtl's formula for the wavelength of a supersonic gas jet. Quart. J. Mech. Appl. Math. **3**, 173–181 (1950).

Consider a steady axisymmetric supersonic jet which emerges from an orifice in the plane $x=0$ with velocity U_1 parallel to the axis of symmetry, with such small excess pressure that shocks are negligible, and with boundary $y\sim 1$ in units of orifice radius $\frac{1}{2}D$. By experiments of R. Emden [Ann. Phys. Chem. (N.S.) **69(305)**, 426–453 (1899)] and others the wave length of the jet is approximately (i) $\lambda=KD(N-1.9)^{\frac{1}{2}}$, where N is the ratio of reservoir to external pressure, while the mean value of $K(N)$ is 0.88 for $N\leqq 10$, 0.94 for $1.9\leqq N\leqq 10$, and 0.78 for $2\leqq N\leqq 20$. For linearized nonviscous flow the velocity potential for the jet is approximately $\phi=\bar{U}x+\sum_1^n A_r J_0(\xi_r y)\sin(\xi_r x/\alpha)$, where $\alpha^2=\bar{M}^2-1$, the Mach number $\bar{M}$ corresponds to mean jet velocity $\bar{U}$, and $J_0(\xi_r)=0$. Terminate the series at $r=1$ and set $\bar{U}=U_1$ to obtain Prandtl's formula [Phys. Z. **5**, 599–601 (1904)]; (ii) $\lambda=1.306D(\bar{M}^2-1)^{\frac{1}{2}}$, which leads to mean $K=1.22$ in (i). The author observes that since U_1 is actually the minimum velocity in the jet, the velocity on the jet boundary is a better approximation to $\bar{U}$. After the A_r have been determined from $\partial\phi/\partial x=U_1$ on $x=0$, the equation $dy/dx\sim\partial\phi/\partial y/\bar{U}$ of the jet boundary yields $y-1=4(1-U_1/\bar{U})\sum_1^n \xi_r^{-2}\sin^2(\xi_r x/2\alpha)$. At the first minimum (iii) $x=\lambda_1=1.22D(\bar{M}^2-1)^{\frac{1}{2}}=2.695D(N^{0.331}-1.205)^{\frac{1}{2}}$, which leads to mean $K=0.94$ for $1.9\leqq N\leqq 10$, 0.83 for $2\leqq N\leqq 20$. Similar investigation of plane jets yields somewhat worse agreement with experiments for $N<8$. This difference between the accuracies of the approximations for axisymmetric and plane jets may be accounted for by the fact that for the same N the maximum displacement $y-1$ of the jet boundary in the former case is less than half of that in the latter, so that the boundary approximations will fail for lower values of N in the plane jet.

J. H. Giese (Havre de Grace, Md.).

Bernard, Jean-J. Sur l'écoulement continu et unidimensionnel à travers une onde de choc droite ou oblique. C. R. Acad. Sci. Paris **230**, 1339–1340 (1950).

The Navier-Stokes equations are applied to determine the internal structure of a shock when the Prandtl number differs slightly from the value $\frac{3}{4}$ for which, as Becker [Z. Physik **8**, 321–362 (1922)] first pointed out, the mathematical problem becomes greatly simplified. It should be noticed, however, that for weak shocks a simple theory was given by Taylor [Proc. Roy. Soc. London. Ser. A. **84**, 371–377 (1910)] for arbitrary Prandtl number, while within stronger shocks the Navier-Stokes equations are too inaccurate for small differences in Prandtl number to be worth taking into account in using them. *M. J. Lighthill.*

Chandrasekhar, S. On the decay of plane shock waves. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. **423**, 14 pp. (1943).

The author shows that, when a plane shock, where the pressure jumps by a factor less than 2.5, moves into still air, both the departure of the velocity of sound behind it from that deduced by neglecting the entropy change, and also the discontinuity in one of the Riemann invariants at the shock, are less than one per cent. He proceeds to deduce the theory of decay of plane shocks, based on the idea of a simple wave combined with use of the Rankine-Hugoniot equations, which, since the paper under review was written, has been published by Friedrichs [Communications on Appl. Math. **1**, 211–245 (1948); these Rev. **10**, 638].

M. J. Lighthill (Manchester).

Chandrasekhar, S. The normal reflection of a blast wave. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. **439**, 11 pp. (1943).

The type of blast wave considered consists of a plane shock followed by a linear decrease of pressure to its atmospheric value. The normal reflection of this wave at a rigid

plane wall is evaluated by the method of characteristics. For simplicity only those shocks which are weak enough for the nonuniformity of specific entropy behind them to be neglected are considered. The Rankine-Hugoniot conditions are applied at the shock surface. In the particular case evaluated the pressure is 1.5 atmospheres behind the incident shock. Immediately after reflection the pressure near the surface rises to 2.2 atmospheres, this being compensated by the smaller extent of the wave as the tail approaches the wall. The strength of the reflected shock at once begins to fall. By the time it has caught up with the tail of incident wave the pressure takes the value 1.52 atmospheres almost uniformly between it and the wall, while the extent of the wave is reduced to slightly less than half its initial value. After this the shock weakens only slightly, but the tail still has to complete its reflection at the wall, after which the pressure is about 1.4 atmospheres behind the shock, falling almost linearly to zero over a slightly greater distance than in the incident wave. *M. J. Lighthill* (Manchester).

Kawamura, Ryuma. On the critical phenomena due to shock waves. Rep. Inst. Sci. Tech. Univ. Tokyo **3**, 43–48 (1949). (Japanese)

Oswatitsch, Kl. Der Verdichtungsstoss bei der stationären Umströmung flacher Profile. Z. Angew. Math. Mech. **29**, 129–141 (1949). (German. English, French, and Russian summaries)

The paper begins with a conventional treatment of weak shock waves in stationary flow, including the calculation of bow- and stern-wave shapes for finite wedges and slender convex contours. It is shown how the drag can be evaluated by computing the entropy rise across the shock waves thus approximated. The identification of this drag with the usual pressure drag is made. The velocity and pressure distributions in the wake can be computed from the entropy distribution. Turning now to slightly supersonic flow, the author assumes irrotational-isentropic flow and obtains approximate formulas for the characteristics and weak shock waves in this region. A rough procedure is suggested for estimating the shock drag of an airfoil with a local supersonic region in a subsonic stream. This does not involve any considerations of boundary-layer separation; nevertheless the author claims that it explains the sharp drag-rise with increasing speed in the transonic region. *W. R. Sears.*

Cabannes, Henri. Sur l'onde de choc attachée lorsque la vitesse aval à la pointe de l'obstacle est subsonique. C. R. Acad. Sci. Paris **230**, 1830–1832 (1950).

In the plane supersonic flow past a sharp-nosed aerofoil, at angles of attack such that the shock wave is attached but the velocity directly behind it on the upper surface at the leading edge is subsonic, the flow conditions near the leading edge are considered. It is shown that the pressure and the streamline deflection depart from their values at this point, to a first approximation, by quantities associated, according to the Prandtl-Glauert laws, with the velocities in the incompressible potential flow in a corner.

M. J. Lighthill (Manchester).

Sanger, Eugen. The gas kinetics of very high flight speeds. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1270**, 49 pp. (1950).

Translation of Zentrale für wissenschaftliches Berichtswesen der Luftfahrtforschung (ZWB), Forschungsber. no. **972** (1938).

Sänger, E. Gaskinetik sehr grosser Flughöhen. Schweiz. Arch. Angew. Wiss. Tech. **16**, 43–63 (1950).

This is a revised version of a previous paper [see the preceding review]. The original paper is perhaps the first investigation of the problem of free molecular flow over convex bodies when the mean free path of the gas is very much larger than the dimension of the body. The forces acting on the body and the energy transfer between the body and the gas can be determined by simple quadrature using the Maxwellian distribution for the molecular velocities and two parameters specifying the collision characteristics of the gas molecules on the solid body: the fraction of specular reflection of the molecules and the accommodation coefficients of the various forms of the molecular energy. The author emphasizes the lack of knowledge of these parameters, especially at higher collision velocities than are usually available from molecular beam methods. Following the author and Bredt [Zentrale für wissenschaftliches Berichtswesen der Luftfahrtforschung (ZWB), Deutsche Luftfahrtforschung Untersuchungen und Mitteilungen no. **3538** (1944), pp. 141–173; translated in Technical Information Branch, BuAer, Navy Department, Translation **CGD-32**] and using classical mechanics, the author tries to estimate the effects of high collision velocity in the excitation of vibrational degrees of freedom and the electron levels, the dissociation, and the ionization. These effects would appear when the velocity exceeds approximately 2000 meters per second. The paper contains a numerical example giving the force coefficients of an inclined plate at radiative equilibrium with the atmosphere. Unfortunately, the mistake made in the original paper on the translational energy associated with the impinging molecules remains uncorrected. This mistake was first pointed out by Stalder and Jukoff [J. Aeronaut. Sci. **15**, 381–391 (1948)]. *H. S. Tsien.*

Westervelt, Peter J. The mean pressure and velocity in a plane acoustic wave in a gas. J. Acoust. Soc. Amer. **22**, 319–327 (1950).

Some results pertaining to second-order effects in one-dimensional travelling waves, predicated by the studies of Airy, Brillouin, and Fubini, are reviewed and correlated. The forces exerted on a stationary sphere situated in a plane progressive wave are calculated. It is found that these forces may have a time-average component considerably greater than, and in opposite direction to, the forces due to radiation pressure. Standing waves of large amplitude are studied from the normal mode point of view. *Y. H. Kuo.*

Spar, Jerome. On the theory of annual pressure variations. J. Meteorol. **7**, 167–180 (1950).

The equations of motion and of continuity are written down in spherical polar coordinates and the atmosphere is assumed to be at rest. This state of rest is perturbed and the perturbation equations integrated in the vertical direction. The resulting equations can be solved provided that a function Q is given, Q depending on the integrated temperature-perturbation throughout the atmosphere. Periodic oscillations of long wave-length are sought for, the function Q being expanded as a series of associated Legendre polynomials in the latitude and of trigonometric functions in the longitude. The coefficients of this series are computed from mean temperature data for January and July. The amplitude of the annual surface pressure oscillation over the whole spherical Earth can thus be found. Charts are given showing the observed and computed surface temperatures and

surface pressures (mean July minus mean January in each case), and agreement is good for large-scale features. Small-scale features are not in very good agreement, chiefly because of the effects of large mountain masses, which were not allowed for in the theory. A subsidiary result is that the effect of friction is probably negligible.

G. C. McVittie (London).

Kiveliovitch, M. Sur la composante verticale du vent. II. J. Sci. Météorologie **2**, 26–31 (1950). (French. English and Spanish summaries)

[For part I see the same J. **1**, 18–22 (1949); these Rev. **11**, 280.] In this paper, the determination of the vertical air velocity from the equations of motion is refined to take into account vertical acceleration. No attempt is made to preserve continuity of mass. *H. Panofsky.*

Fleagle, Robert G. A theory of air drainage. J. Meteorol. **7**, 227–232 (1950).

Neglecting the effect of the earth's rotation, the equation of motion is derived for a compressible fluid cooling at the bottom by contact with a radiating surface of uniform slope and large extent. If friction is assumed directly proportional to the velocity, the resultant differential equation is second order in the drainage velocity, and the usual periodic solution varies about an equilibrium value which is directly proportional to the net outgoing radiation from the ground and inversely proportional to the height to which the cooling extends and to the (constant) slope of the ground. Compressibility of the fluid plays a major role, since it is the adiabatic warming due to downslope motion which counteracts the radiational cooling and produces the periodicity. A nonperiodic solution neglecting acceleration is also obtained for the case in which friction is assumed proportional to the square of the drainage velocity. The result shows that the rate at which equilibrium is attained is dependent on the assumed form of the frictional force.

W. D. Duthie (Monterey, Calif.).

Davies, D. R. Three-dimensional turbulence and evaporation in the lower atmosphere. I. Quart. J. Mech. Appl. Math. **3**, 51–63 (1950).

La théorie d'évaporation d'un liquide saturé d'une surface plane dans un courant d'air turbulent a été étudié par Sutton [Proc. Roy. Soc. London. Ser. A. **146**, 701–722 (1934)] qui a trouvé des résultats, pour le transfert vertical, qui s'accordent assez bien avec les observations. Malheureusement ses résultats en ce qui concerne la distribution de la vapeur en aval du vent, ne sont pas satisfaisants. Pour parer à cette difficulté l'auteur a introduit, dans une recherche antérieure [Proc. Roy. Soc. London. Ser. A. **190**, 232–244 (1947); ces Rev. **8**, 611] une diffusion latérale proportionnelle à une puissance simple de l'altitude. On obtient une solution en se donnant une distribution de la vapeur sur une surface plane limitée par une parabole. Il a été obligé de supposer que cette aire parabolique s'étend indéfiniment en aval du vent, ce qui exclut la possibilité d'obtenir la distribution de la vapeur en aval du vent pour une aire finie. Dans le mémoire actuel l'auteur se propose de lever cette difficulté. Il considère l'évaporation d'un bassin rectangulaire, le courant étant dirigé parallèlement à un des côtés du rectangle. En première approximation on suppose que la vitesse du vent et la diffusion sont constantes. Après quelques transformations des variables, l'auteur obtient une solution en utilisant les fonctions de Mathieu. La distribution de la vapeur en aval du vent est donnée par analogie avec un problème de la conduction de la chaleur. L'auteur étudie ensuite le cas ou la vitesse du vent et la diffusion sont fonctions des puissances simples de la hauteur. Il obtient une solution complète du problème qu'il se propose de comparer aux observations dans une deuxième partie du travail [voir l'analyse ci-dessous]. *M. Kiveliovitch* (Paris).

Davies, D. R. Three-dimensional turbulence and evaporation in the lower atmosphere. II. Quart. J. Mech. Appl. Math. **3**, 64–73 (1950).

Dans la première partie [voir l'analyse ci-dessus], l'auteur a obtenu une solution analytique du problème en supposant que les coefficients de diffusion et la vitesse du vent sont des puissances simples de l'altitude. Dans cette partie de son travail, l'auteur essaie de déterminer les valeurs numériques de ces constantes en utilisant des théories récentes bidimensionnelles, de Sutton [Quart. J. Roy. Meteor. Soc. **73**, 257–281 (1947)] et Calder [same J. **2**, 153–176 (1949); ces Rev. **11**, 66], et discute en détail ces résultats en les comparant aux observations. *M. Kiveliovitch* (Paris).

Elasticity, Plasticity

Stakgold, Ivar. The Cauchy relations in a molecular theory of elasticity. Quart. Appl. Math. **8**, 169–186 (1950).

Bekannterweise kann man die potentielle Energie von einem Volumenelement eines elastisch deformierten Körpers (das elastische Potential) folgendermassen schreiben

$$\Phi = \Phi_0 + \Omega^{ij}e_{ij} + \Lambda^{ij,kl}e_{ij}e_{kl} + \cdots.$$

Das absolute Glied verschwindet jedoch und, wenn die ursprünglichen äusseren Kräfte gleich Null sind, auch das in den Deformationskomponenten (e_{ij}) lineare Glied. Es bleibt also nur das quadratische Glied mit seine 81 Tensorkomponenten $\Lambda^{ij,kl}$ übrig. Aus der blossen Definition der Deformationskomponenten folgt jedoch, dass sich die auf 36 reduzieren und aus dem Vorhandensein des elastischen Potentials folgt eine weitere Reduktion auf 21. Cauchy hat weiter gezeigt, dass, wenn zwischen den Molekülen nur Zentralkräfte wirken, man ausserdem noch sechs Relationen zwischen den Komponenten herleiten kann. In der vorliegenden Arbeit werden die zur Gültigkeit der Cauchyschen Zusammenhänge notwendigen und hinreichenden Bedingungen besprochen. Bei einem aus nur einer Atomart aufgebauten Gitter wird der schon von Voigt und Born angegebene Gedankengang mathematisch streng durchgeführt, wobei nur die Annahme eingeführt wird, dass es solche kleine Gebiete gibt, in denen die Deformation noch als homogen betrachtet werden kann, die jedoch gross im Verhältnis zur Reichweite der zwischen den Atomen wirkenden Kräfte sind. Bei aus verschiedenen Atomarten aufgebauten Gitter wird die schon von Born festgestellte Tatsache besprochen, dass bei denen auch im Falle von Zentralkräften die Cauchyschen Beziehungen nicht mehr gültig sind. *T. Neugebauer* (Budapest).

Tolotti, Carlo. Sulla statica delle superficie sviluppabili inestendibili ed elasticamente flessibili. Giorn. Mat. Battaglini (4) **3**(79), 1–48 (1950).

The author continues his researches in the theory of elastically flexible but inextensible membranes for which he has derived the general equations [Atti Accad. Naz. Lincei.

Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 369–374, 605–609 (1946); same Giorn. (4) 2(78), 128–150 (1949); these Rev. 8, 359; 11, 486]. In the present paper he obtains the general solution of his equations for the case of a developable surface, generalizing the solutions given by Laura [Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 99, 339–356 (1940)] for Beltrami's equations (in which the membrane is supposed to be without flexural elasticity). He shows that if the deformed configuration Γ of a curve upon the undeformed membrane is prescribed, then the rectifying developable of Γ is an equilibrium configuration for the membrane. As might be expected, however, it is generally not possible to find a solution corresponding to prescribed forces and moments upon the boundary. The author employs a variational principle to obtain suitable boundary conditions in terms of forces, and these he discusses and interprets. Finally, he transforms the equation into another form suitable for attack by the direct method of the calculus of variations, to which he promises to devote a subsequent paper. *C. Truesdell* (Bloomington, Ind.).

Yoshimura, Yoshimaru. Theory of thin shells with finite deformation. I. Rep. Inst. Sci. Tech. Univ. Tokyo 2, 167–173 (1948). (Japanese)

Yoshimura, Yoshimaru. Theory of thin shells with finite deformation. II. Rep. Inst. Sci. Tech. Univ. Tokyo 3, 19–25 (1949). (Japanese)

Yoshimura, Yoshimaru. A method of successive approximation of the behaviour of a compressed rectangular plane plate after buckling. Rep. Inst. Sci. Tech. Univ. Tokyo 2, 131–137 (1948). (Japanese)

Yoshimura, Yoshimaru, and Uemura, Masuji. Vibrations of a pressurised thin circular cylindrical shell. Rep. Inst. Sci. Tech. Univ. Tokyo 3, 117–121 (1949). (Japanese)

Koepcke, W. Zur Ermittlung der Einflussflächen und inneren Kräfte umgangsgelagerter Rechteckplatten. Ing.-Arch. 18, 106–138 (1950).

The author outlines procedure, and also provides extensive numerical tables to facilitate calculations in practice, for solution of deflection and stress problems for statically loaded rectangular plates with the four edges restrained in arbitrary combinations of simple and built-in supports. The general manner of solution is as follows: (a) a particular solution is found for the nonhomogeneous differential equation for plate deflection, corresponding to the actual plate loading but not necessarily satisfying the edge boundary conditions; (b) through the addition of a biharmonic polynomial to solution (a), the conditions at the four plate corners are satisfied; (c) each edge is then treated in succession, according to the scheme: a solution for an unloaded plate is derived, with the actual boundary conditions of the problem satisfied at all edges except the one in question; here the boundary conditions are arbitrary. This solution is added to (a) and (b) and the proper boundary condition at the edge in question is imposed on the total solution. In the general case, a solution of type (c) must be added for each edge.

By means of tables, the author systematizes the procedure for introducing the solutions (c) into a particular problem. When the two edges adjacent to the edge in question are simply supported, the procedure for both symmetric and anti-symmetric plate loadings is relatively direct. When this is not the case, because of the unavailability of a precise solution of type (c), further difficulties are encountered. For the latter circumstance, the edge boundary conditions are not satisfied exactly, but are satisfied by a requirement derived from the condition that the sum of the squares of all the boundary condition errors shall be a minimum. Two typical problems are worked out in detail to illustrate the application of the procedure.

M. Goland (Kansas City, Mo.).

Uflyand, Ya. S. On a case of bending of a rectangular plate. Doklady Akad. Nauk SSSR (N.S.) 72, 655–657 (1950). (Russian)

Consider a thin rectangular plate which is subjected to a normal load and which is clamped at three edges and simply supported at the fourth. The deflection is given as a series expansion in trigonometric and hyperbolic functions containing as unknowns the load distributions on the two opposing clamped edges. (Convergence of this series is not discussed.) If the load on the plate is symmetric with respect to a line midway between these edges, these load distributions are given by the solution of a mixed Fredholm-Volterra integral equation. Approximate numerical values of the maximum shearing force and bending moment in the plate are given for three types of symmetric loading.

H. I. Ansoff (Santa Monica, Calif.).

Fridman, M. M. The flexure of a thin isotropic rectangular plate with a soldered-on isotropic washer of another material. Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 429–432 (1950). (Russian)

O'Chou, Chang. Two dimensional theory of stiffened plates. Aeronaut. Quart. 2, 1–8 (1950).

A theory for the stress distribution in a stiffened plate subjected to edge tractions is presented. Stiffeners are considered to be attached to the plate in two perpendicular directions. It is assumed that the axial load carrying properties of the stiffener can be uniformly distributed. Using this assumption, it is found that the governing differential equation is similar in form to the customary equation $\nabla^4\varphi=0$ for an unreinforced sheet. Solutions of the basic equation are presented for two cases: one, a rectangular stiffened plate under pure bending in its plane; and the other, a semi-infinite stiffened plate under a concentrated force acting perpendicular to the boundary. *S. Levy.*

Müller-Magyari, F. Beiträge zur Zugfeldtheorie dünnwandiger Plattenstreifen. I, II. Österreich. Ing.-Arch. 4, 12–27 (1950).

The buckling and post-buckling behavior of a thin rectangular plate under shear, longitudinal, and cross compressive stresses is analyzed in part I. With the aid of some simplifying assumptions the author is able to determine the stresses for transition from the cylindrical buckling form to the more usual diagonal wave pattern up to loads 25 times the Euler critical. Part II deals with shear buckling of an orthotropic long thin rectangular plate of any orientation of grain. *D. C. Drucker* (Providence, R. I.).

Gol'denveizer, A. L. On the theory of thin-walled bars. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 561–596 (1949). (Russian)

Consider an elastic thin-walled cylindrical bar of arbitrary cross-section with varying wall thickness. The lateral load R is arbitrary but similar at all cross-sections and the

end forces and moments are prescribed. The problem is to determine that part of the complete stress distribution which can be used with sufficient accuracy to represent the stress state in the bar as the distance from the ends is increased. Thus the rapidly decaying local end stress distributions are neglected. This slowly decaying stress distribution is referred to as the fundamental stress state. It is shown that if and only if the load R varies linearly along the length of the bar, it can be separated into two components one of which is statically equivalent to R at each cross-section and the other is self-equilibrated at each cross-section. Under these conditions simple solutions of the problem can be constructed for each component, the first using momentless shell theory and the second treating each section as a simple arc. Next, solutions are determined for an unloaded bar which satisfy the end conditions. Among these solutions only those are retained which decay slowly along the length of the bar. Limits are determined on the length of the bar within which the present results are applicable. Above the upper limit, thin-walled bars begin to behave just like solid bars and the classical theory is applicable. Below the lower limit, the local stress distributions can no longer be neglected and the present theory does not apply. Within these limits the fundamental stress state can be described by a linear combination of the two sets of solutions provided the end moments and forces are replaced by appropriate integral conditions.

H. I. Ansoff (Santa Monica, Calif.).

Deschodt, J. Calcul des déformations d'un arc encastré à une extrémité. Recherche Aéronautique **1950**, no. 16, 19–25 (1950).

The general integrals for the deflection and rotation of a section of a curved rod, built in at one end and subjected to forces and couples at the other end, are written down, with the usual assumptions for small displacements of thin rods of arbitrary, slowly varying section. Special cases are considered when the mean fiber of the rod initially forms a plane curve which determines a plane of symmetry, when force and moment components in and perpendicular to this plane are applied individually, and when the initial curve is a circular arc. *E. H. Lee* (Providence, R. I.).

Garavaldi, Orestina. Sul problema del "fioretto di perforazione" trattato col metodo operazionale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8**, 365–368 (1950).

The solution of the physical problem by operational methods leads to (*) $[(b+\tanh\varphi)/(1+b\tanh\varphi)+a/\Delta]^{-1}1(t)$, where a, b are constants, Δ is the operator d/dt, φ is a constant multiple of Δ, and $1(t)$ is Heaviside's unit function. The author gives a series expansion of the function (*).

A. Erdélyi (Pasadena, Calif.).

Federhofer, Karl. Über die Biegungs-Drillungsschwingungen des Kreisringes mit doppelt-symmetrischem Querschnitte. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **157**, 299–320 (1950).

For a circular ring with a principal axis of the cross-section in the ring plane, the normal vibrations involving cross-section twist and bending normal to the ring plane are uncoupled from other mode types. For this mode, the author derives equations of motion for a ring with double-symmetrical cross-section, and develops the characteristic frequency equation for the square cross-section case. One-dimensional bending and torsion theory is used, but with such effects as rotatory and warping inertia, and shear and warping energy taken into account. These secondary effects, which are conventionally omitted from the analysis, permit the author to show good analytic agreement with frequency measurements made by W. Kuhl [Akustische Z. **7**, 125–152 (1942)] for a wide range of lower- and higher-order modes.

M. Goland (Kansas City, Mo.).

Buckens, F. Influence of the relative radial thickness of a ring on its natural frequencies. J. Acoust. Soc. Amer. **22**, 437–443 (1950).

The author discusses the problem of a ring vibrating in its plane using the theory for the bending of thick curved bars and also taking account of direct shearing. Cases of rings having rectangular and elliptical cross-sections are discussed in detail, and it is found that the effect of shearing on the natural frequencies is greater than that of other effects, such as rotational inertia and curvature. The shearing effect is found to be of greatest importance in the higher modes of vibration. The influence of extensibility is then considered and its effect on the frequencies is found to be small, particularly for the higher modes. The effects of shearing and rotational inertia on the vibration of straight beams are then compared with those on the vibration of rings. *H. D. Conway* (Ithaca, N. Y.).

Kostyuk, A. G. On the equilibrium of an annular plate with a power law of hardening. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 319–320 (1950). (Russian)

Stress distribution in a thin annular plate is determined under the following assumptions. The thickness is given by $h=h_0\rho^{-\lambda}$, where ρ is the normalized radius and λ is a constant. Radial stress distribution is prescribed at the circular edges, stress normal to the plate is set equal to zero, and the stress distribution across the thickness is assumed uniform. The hardening law is given by $\sigma_i=N\epsilon_i^{\mu}$ where σ_i and ϵ_i are the stress and the strain intensities, respectively, N is a constant, and μ is a parameter in the range $0\leqq\mu\leqq 1$. A graph presents the normalized radial and tangential stress distributions in the plate. The solution is next modified to give the bending moment distribution in a plate subjected to uniformly distributed bending moments at the boundary.

H. I. Ansoff (Santa Monica, Calif.).

Pearson, C. E. Bifurcation criterion and plastic buckling of plates and columns. J. Aeronaut. Sci. **17**, 417–424, 455 (1950).

The stability of nonconservative systems is considered, in particular, buckling with plastic flow. Assuming an initial displacement at a load less than the tangent modulus load, the behaviour on subsequent loading is considered. An idealised strut model, a strut, and a plate in compression are considered. For small deflections all compressive stresses increase, subsequently certain regions unload. The preferred path is considered that which is approached as the initial displacement approaches zero. This limit is not the undeflected configuration. The plate results are compared with experimental and earlier theoretical results. *E. H. Lee.*

Wolf, H. The propagation of torsional plastic waves in circular cylindrical tubes and shafts. Graduate Division of Applied Mathematics, Brown University, Providence, R. I., Tech. Rep. **A11–50**, 41 pp. (1949).

In the reduction of the three-dimensional equations of small motions of a continuum to special forms applicable

to longitudinal and transverse motions of bars, it is customary to neglect terms which are important for wave lengths short in comparison with the lateral dimensions of the bar. In the case of torsional motions of a bar or tube of circular cross section it is not necessary to neglect such terms in order to arrive at a single, second order, partial differential equation of motion. Solutions of this equation may be expected to conform more closely with impact experiments than solutions of the corresponding equations for longitudinal and transverse motions. The author considers the case of torsional motions and shows that, whereas for an elastic medium the final equation is linear and contains only one independent space variable, for a plastic medium the equation is nonlinear and contains two independent space variables. Nevertheless he is able to obtain useful solutions, by a method of successive approximations, for a reasonable deformation law of plasticity and appropriate boundary conditions. *R. D. Mindlin* (New York, N. Y.).

Zverev, I. N. The propagation of a disturbance in a visco-elastic and visco-plastic bar. Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 295–302 (1950). (Russian)

Longitudinal wave propagation in a visco-elastic bar (stress-strain relation $\sigma=E\epsilon+\mu\dot{\epsilon}$) is studied using Laplace transform methods. A semi-infinite bar with given end velocity or end strain variation is solved. Impact by rigid mass is also considered. The solution is modified for a visco-plastic bar ($\sigma=\sigma_s+E'(\epsilon-\epsilon_s)+\mu\dot{\epsilon}$). *E. H. Lee* (Providence, R. I.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Violette. La condition des sinus pour une lentille épaisse. Rev. Optique **29**, 259–264 (1950).

The author determines the form of an aplanatic lens, i.e., a lens consisting of two spherical surfaces such that, for a given magnification, spherical aberration and sine condition are corrected. The author obtains two differential equations, which he claims can be integrated by an approximation method due to Adams. *M. Herzberger.*

Temmerman, Frits. Une généralisation de la formule de Kerber. Rev. Optique **29**, 195–199 (1950).

Differencing paraxial ray tracing formulae from those of finite rays, the author obtains a general difference formula which contains Kerber's difference formula and the formulae of Staeble and Lihotzky as special cases. The corresponding formulae can be brought into a form suggested by T. Smith for a particular case in which it can be applied as a control formula for tracing rays through an optical system.

M. Herzberger (Rochester, N. Y.).

Swings, P. Remarques sur les systèmes catadioptriques de Bouwers. Bull. Soc. Roy. Sci. Liège **19**, 168–176 (1950).

The paper contains a discussion of systems combining a mirror with a meniscus. An historic introduction mentioning the contribution of Bouwers and Maksutov is followed by a series of examples. The author discusses first a system where the two surfaces of the accessory lens are concentric with the mirror; then he adds to it another concentric mirror to obtain a system of Cassegrain type. A short discussion of a system where the meniscus lens is not concentric with the mirror is followed by the discussion of a system consisting of a mirror and a plane parallel plate. The paper does not contain the complete algebraic theory of the systems discussed, but gives a series of practical examples. [With respect to the historical note the reviewer should like to mention the fact that Maksutov's aim was to correct color, whereas Bouwers and the author aim primarily at the correction of spherical aberration.] *M. Herzberger.*

Matukuma, Takehiko. Theory of Schmidt camera. Preliminary report. Sci. Rep. Tôhoku Univ., Ser. 1. **33**, 43–50 (1949).

After an historical introduction the author gives without derivation formulae for the correcting surface in a Schmidt camera, in which the focal length is slightly different from $\frac{1}{2}R$ (R being the mirror radius). The coordinates of the equation of the curve can be developed with respect to a well chosen parameter t, and the author states that the problem of convergence has been investigated by him. The author discusses the sine condition, chromatic aberration, as well as the trace of oblique rays through the Schmidt camera. One has to wait for the promised publication to understand the details of this paper. [The reasoning in the right column of page 45 with respect to the smallness of the sine condition does not seem to be quite convincing to the reviewer.] *M. Herzberger* (Rochester, N. Y.).

Schlögl, Josef. Zur Elektronenoptik magnetischer Linsen von Glaserschem Typus. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **157**, 237–262 (1950).

The objective is the practical evaluation of the electron trajectory of paraxial electrons moving in an axially symmetrical magnetic field with a distribution along the axis $H(z)=H_0[1+(2/a)^2]^{-s}$ as formulated by Glaser [Z. Physik **117**, 285–315 (1941); these Rev. **4**, 32]. In order to approximate the results of experimental measurements, only the values of $\mu=1, 3/2, 2, 5/2, 3, \cdots$ are of practical importance. Using $2(\mu-1)=m$, the equation for the electron trajectory can be transformed into

$$(1)\qquad \frac{d^2v}{d\varphi^2}+(1+k^2\sin^{2m}\varphi)v=0,$$

an equation of the Hill type; the variable φ with $0\leqq\varphi\leqq\pi$ is defined by $z/a=\cot\varphi$, whereas $v=(r/a)\sin\varphi$ with r the radial distance from the axis. The constant a is determined from the measured distribution curve of $H(z)$, and $k^2=eH_0^2a^2/8mU$, where e/m denotes the ratio of charge to mass of the electron, U its acceleration voltage equivalent to its initial velocity. Because of the lack of a fundamental function system, (1) is rewritten as $L(v)=-k^2(\sin^{2m}\varphi-1)v$, where $L(v)$ is the self-adjoint differential expression

$$(d^2v/d\varphi^2)+(1+k^2)v.$$

If u is a solution of $L(u)=0$, a Green's function $G(\varphi,\xi)$ can be constructed which by means of Green's relation leads to the inhomogeneous Fredholm integral equation

$$(2)\qquad v(\xi)=C\frac{\sin\omega\xi}{\sin\omega\pi}+k^2\int_0^\pi G(\varphi,\xi)(\sin^{2m}\varphi-1)v(\varphi)d\varphi,$$

where $\omega^2=1+k^2$ must not be an integer. Expanding $G(\varphi,\xi)$ into a Fourier series by selecting the fundamental interval $0\leqq\varphi\leqq\pi$ as half period of an odd function brings (2) into a convergent series form. It is stated that the conventional

methods of iterated kernels does not lead to practical results. The simplest approach to a solution is the use of a finite number of the terms and determination of the coefficients $a_\nu = \int_0^\pi \sin \nu\xi (\sin^{2m}\xi - 1)v(\xi)d\xi$ from a set of linear algebraic equations. This leads to the final solution of (2) in the form

$$v(\varphi) = C\frac{\sin \omega\varphi}{\sin \omega\pi} + \frac{2}{\pi}k^2 \sum_{\nu=1}^{N} a_\nu g_\nu(\varphi),$$

where $g_\nu(\varphi) = (\nu^2 - \omega^2)^{-1} \sin \nu\varphi$. Comparison for $m=2$ and choice of $N=5$ with a solution given by Svartholm [Ark. Mat. Astr. Fys. **28B**, no. 16 (1942)] in form of an infinite series shows excellent numerical agreement for the focal length.

If $\omega^2 = 1 + k^2 = n^2$, an integer, then the homogeneous boundary value problem $L(u) = 0$ with $u(0) = u(\pi) = 0$ has $u_0 = (2/\pi)^{\frac{1}{2}} \sin u\varphi$ as solution and an expanded form of Green's function is required. This is found as solution of the inhomogeneous problem $L(u) = u_0(\varphi)u_0(\xi)$ and transformation is then made to the integral equation as before. The general solution is indicated and an example given for $k^2 = 3$. In the second part of the paper, the reverse problem is treated, namely to find the eigenvalues of k^2 for a specified focal length by means of the Rayleigh-Ritz variational method. *E. Weber* (Brooklyn, N. Y.).

Toraldo di Francia, G. Principi variazionali per i coefficienti di riflessione delle barriere de potenziale. Nuovo Cimento (9) **7**, 255–265 (1950).

A detailed account of the author's recent note [Physical Rev. (2) **78**, 298 (1950); these Rev. **11**, 760].

A. E. Heins (Pittsburgh, Pa.).

Calamai, G. Sulla non uniforme trasparenza dei reticoli. Ottica (N.S.) **4**, 21–28 (1950).

The Fresnel-Kirchhoff diffraction formula is used to obtain an approximate solution of the problem of diffraction by a grating of nonuniform transparency.

E. T. Copson (Dundee).

Woonton, G. A. A more exact Fresnel field diffraction relation. Canadian J. Research. Sect. A. **28**, 120–126 (1950).

The author points out that the application of Kirchhoff's formula to diffraction problems, despite its inconsistencies, is successful in predicting the Fraunhofer field of practical radiators as used in micro-wave technique. He emphasizes the need for similar approximations in the Fresnel zone. The latter are derived from the Fraunhofer approximation on the assumption that (1) the original aperture distribution is linear, that is, $\varphi(x, y) = F_1(x)F_{11}(y)$, and (2) the resulting distribution in any plane parallel to the aperture in the Fresnel zone is linear, $\varphi'(\xi, \eta) = F_2(\xi)F_{22}(\eta)$. The functions F_2 and F_{22} are expressible as Bessel function transforms of F_1 and F_{11} respectively. [The author does not justify the assumption (2).] *C. J. Bouwkamp* (Eindhoven).

Woonton, G. A. The effect of an obstacle in the Fresnel field on the distant field of a linear radiator. J. Appl. Phys. **21**, 577–580 (1950).

The radiation field from a linear radiator is obstructed by a second aperture placed a short distance in front of it and at an arbitrary angle to its normal. The author develops formulas for the distant field produced by the combination. The author's analysis is mainly a repetition of earlier work [see the preceding review]. *C. J. Bouwkamp.*

Artmann, Kurt. Beugung polarisierten Lichtes an Blenden endlicher Dicke im Gebiet der Schattengrenze. Z. Physik **127**, 468–494 (1950).

In the discussion of diffraction of light on a cylinder one encounters the series

$$\tfrac{1}{2}\sum_{n=0}^{\infty} \{H_n^2(kr) - H_n^2(ka)H_n^1(kr)/H_n^1(ka)\}e^{in(\varphi+\pi/2)}$$

and a second similar series. In the case which is of interest to the author ka and kr are both large, and kr is large in comparison with ka. It turns out that the most important terms are those for which n is approximately equal to ka. For these terms the Hankel functions with variable kr may be replaced by their asymptotic expressions. After this replacement the author approximates the series by an integral with respect to n, and evaluates the integral approximately for small φ (near the boundary of the geometrical shadow of the cylinder). The results are used for a discussion of the physical problem. *A. Erdélyi* (Pasadena, Calif.).

Papas, Charles H. Diffraction by a cylindrical obstacle. J. Appl. Phys. **21**, 318–325 (1950).

A plane wave is incident upon an infinite perfectly conducting cylinder. The electric vector of the plane wave is parallel to the generators of the cylinder. By employing variational methods similar to those employed by Levine and Schwinger [Physical Rev. (2) **74**, 958–974 (1948); (2) **75**, 1423–1432 (1949); these Rev. **10**, 221, 764], the author calculates the total plane-wave scattering cross section. Numerical results are provided. *A. E. Heins.*

Aoki, Tosio, Sunouchi, Gen-ichirô, and Utagawa, Masatomo. On the diffraction of electromagnetic waves. Sci. Rep. Tôhoku Univ., Ser. 1. **33**, 38–42 (1949).

The authors discuss the Babinet principle and makes several applications to electromagnetic and acoustic diffraction. *A. E. Heins* (Pittsburgh, Pa.).

Meixner, J., und Andrejewski, W. Strenge Theorie der Beugung ebener elektromagnetischer Wellen an der vollkommen leitenden Kreisscheibe und an der kreisförmigen Öffnung im vollkommen leitenden ebenen Schirm. Ann. Physik (6) **7**, 157–168 (1950).

Until recently, the problem of the diffraction of a plane-polarized electromagnetic wave by a perfectly conducting infinitely thin circular disk had not been solved in a satisfactory manner, despite numerous attempts by various authors. It has now become clear that most of them failed because they did not recognize the correct singularities on the rim of the disk. In the paper under review a remarkably simple solution is presented. Observing that the diffracted field is derivable from the currents induced in the disk by the incident wave, the authors express the diffracted field in terms of the Hertzian vector potential which obviously is everywhere parallel to the plane of the disk. The two rectangular components of the vector potential are expanded in oblate spheroidal wave functions. The coefficients in these expansions are derived from the boundary conditions on the disk and its rim in a unique way. For normal incidence, the final solution only involves spheroidal wave functions of order 0 and 2. Special attention is drawn to the long-wave limiting case, in which the first few terms of a power series in ka are derived. An approximate expression is given for the scattering coefficient of a small circular disk for the case of normal incidence. The complementary problem (diffraction by a circular hole) is solved by application of Babinet's

principle. Numerical solutions are promised in a subsequent paper. (The footnote reference 13 should read J. W. Miles, J. Appl. Phys. **20**, 760–761 (1949)) [these Rev. **11**, 141].
C. J. Bouwkamp (Eindhoven).

Mirimanov, R. G. On the solution of a general problem of applied electrodynamics. Doklady Akad. Nauk SSSR (N.S.) **71**, 879–882 (1950). (Russian)

The problem is that of a homogeneous medium containing sources and also homogeneous diffracting bodies of arbitrary shape. Starting from methods of Stratton and Chu [Physical Rev. (2) **56**, 99–107 (1939)] the author finds the following integral equation for the resultant field:

$$\mathbf{E}(r)=\mathbf{E}_0(r)+(4\pi)^{-1}(\mu_0 k_a^2/\mu_a k_0^2-1)\int(\mathbf{n}\cdot\mathbf{E}(r'))\,\mathrm{grad}\,\phi\,dS,$$

with a corresponding magnetic equation, where $\mathbf{E}_0(r)$ is the primary field and ϕ has the form e^{ikR}/R. The surface integral is taken over the surfaces of separation and can, according to the author, be interpreted as a dipole distribution. There are applications to the cases of an infinite conducting plane and of a thin unclosed shell of general shape. [Reviewer's comments. It is not apparent whether the author is using the same or different ϕ's for the various regions. The point is made clear by C. Müller [Arch. Math. **1**, 296–302 (1949), p. 301; these Rev. **11**, 293]; cf. also W. von Ignatowsky [Ann. Physik (4) **18**(323), 495–522 (1905), p. 505]. Furthermore, the surface integral forming the secondary field has the form of a gradient, and cannot represent any appropriate electromagnetic field.] *F. V. Atkinson.*

Bailey, V. A., and Roberts, J. A. Graphical study of the dispersion of electro-magneto-ionic waves. Australian J. Sci. Research. Ser. A. **2**, 307–321 (1949).

In the theory of plane-wave propagation through an ionized medium under influence of static magnetic and electric fields [cf. Bailey, same J. Ser. A. **1**, 351–359 (1948); these Rev. **11**, 562], the equation of dispersion is in general an irreducible equation of the eighth degree both in the frequency and the wave number. The authors describe a graphical method for approximately solving this equation in various cases of practical importance. Three physical aspects of the dispersion equation are considered according as the phase velocity, the frequency, or the wave number is given real. Several curves are shown of wave number versus frequency. *C. J. Bouwkamp* (Eindhoven).

Macfarlane, G. G., and Hay, H. G. Wave propagation in a slipping stream of electrons: small amplitude theory. Proc. Phys. Soc. Sect. B. **63**, 409–427 (1950).

The slipping stream of electrons consists of an electron beam moving parallel to the z-axis with a velocity which is a function of one transverse coordinate y. Such a stream may be obtained by applying crossed electric and magnetic fields. In the nonoscillatory state the motion of the electrons may be obtained from an action function ψ_0, and for small oscillations a similar function $\psi=\psi_0+\psi_1\exp(\omega t-hz)$ is assumed. Retaining only first order perturbation terms a differential equation for ψ_1 is obtained, and its solution for linear velocity variation across the stream discussed. For an electron stream in free space between unperturbed boundaries $y=\pm a$, the wave propagation constant h is determined by matching the wave admittance at the perturbed boundary of the stream to the admittance of the external field. It is found that the variation of h depends on the magnitude of the velocity parameter $\alpha=(V_a-V_{-a})/2V_0$. When $\alpha<0.42$, three different types of amplifying wave may be recognized. The principal one acts like the dominant amplifying wave in a two-beam tube with electron velocities V_{-a} and V_{+a}, while the remaining two resemble the two dominant waves in a traveling-wave tube. On the other hand, when $\alpha>0.42$ these two waves become dominant, one of them absorbing the two-beam wave. The gain for each type of wave is evaluated, and it is found that the slipping stream type has lower maximum rate of gain than either the two-beam tube or the traveling-wave tube. The slipping stream may also be used inside a slow-wave TM waveguide. In this case it is found that the wave propagation has characteristics similar to those of the traveling-wave tube. The propagation constant is derived as a function of the ratio of the wave velocity in the empty guide to the maximum electron velocity, and it is shown that maximum gain is obtained when this ratio is approximately unity.
M. C. Gray (Murray Hill, N. J.).

Schumann, Winfried Otto. Über den Durchgang elektromagnetischer Wellen durch Plasmaschichten endlicher Dicke unterhalb der Grenzfrequenz. Arch. Elektr. Übertragung **4**, 173–174 (1950).

Bekannterweise werden auf eine unendlich ausgedehnte homogene Plasmaschicht (die also freie Elektronen enthält) senkrecht einfallende elektromagnetische Wellen unterhalb der kritischen Kreisfrequenz $(2\pi\nu_0)^2=\omega_0^2=Ne^2/\epsilon_0 m$ total reflektiert, wo ϵ_0 die Dielektrizitätskonstante des Vakuums in technischen Einheiten bedeutet und alle anderen Buchstaben die gewohnte Bedeutung haben. In dieser Arbeit wird der Fall behandelt, dass die Plasmaschicht nicht unendlich ausgedehnt, sondern blos eine verhältnismässig dünne planparallele Schicht ist, und es folgt mit Hilfe der üblichen Stetigkeitsbedingungen für die Amplitude der doch hindurchtretenden Welle

$$A_3=\frac{2A_1\exp(i\delta\omega/c)}{2\cosh(\delta\epsilon_p'^{\frac{1}{2}}\omega/c)-i(\epsilon_p'^{\frac{1}{2}}-\epsilon_p'^{-\frac{1}{2}})\sinh(\delta\epsilon_p'^{\frac{1}{2}}\omega/c)},$$

wo A_1 die Amplitude der einfallenden Welle, δ die Schichtdicke, ω die Kreisfrequenz ($\omega<\omega_0$) und $-\epsilon_p'=\epsilon_p$ die Dielektrizitätskonstante der Plasmaschicht bedeuten. Der Reflexionskoeffizient der Plasmaschicht und der Energietransport im Plasma werden ebenfalls berechnet und einige spezielle Fälle werden besprochen. Das mathematische Problem ist zu dem bei dem Durchgang des Lichtes durch eine dünne "totalreflektierende" Schicht auftretenden und zu dem quantenmechanischen Problem des Durchganges von geladenen Teilchen durch einen Potentialberg (Tunneleffekt) vollständig analog. *T. Neugebauer* (Budapest).

Ortusi, J., et Simon, J. C. Le principe de conservation de l'énergie et les formules de Kottler. Ann. Radioélec. **5**, 67–73 (1950).

Some physical remarks about the radiation and reflection properties of a semi-infinite rectangular wave guide.
A. E. Heins (Pittsburgh, Pa.).

Leitner, Alfred, and Spence, R. D. Effect of a circular groundplane on antenna radiation. New York University, Washington Square College, Mathematics Research Group, Research Rep. no. **EM-19**, i+36 pp. (1950).

The authors investigate the radiation properties of an antenna system consisting of a quarter-wave dipole antenna above a circular conducting disk of zero thickness. The

system is driven by a point generator between the centre of the disk and the base of the dipole, which is perpendicular to the disk and extends along the Z-axis from $z=0$ to $z=\frac{1}{4}\lambda$. The current in the dipole is assumed to be $I=I_0\cos kz$, where $k=2\pi/\lambda$ is the wave number. The field of the [hypothetical] isolated dipole is known from theory [cf. J. A. Stratton, Electromagnetic Theory, McGraw-Hill, New York, 1941, pp. 454–457]. The problem solved by the authors is that of diffraction of this primary field by the circular disk. The field of the antenna system is completely fixed by the azimuthal component of the magnetic vector H_φ which is independent of φ since the system has axial symmetry about the dipole axis. The function H_φ is expanded in oblate spheroidal wave functions of the first order [cf. J. A. Stratton, P. M. Morse, L. J. Chu, and R. A. Hutner, Elliptic Cylinder and Spheroidal Wave Functions . . . , Wiley, New York, 1941; these Rev. 4, 281]. The coefficients in this expansion follow from the assumed line source and the boundary condition $\partial H_\varphi/\partial z=0$ on the disk. The radial currents on either side of the disk are calculated as well as the distant field of the antenna. The absolute value of the total current crossing the rim of the disk is $\frac{1}{2}I_0$ and is independent of the radius a of the disk. [It is to be noted that the sum of the currents on the two sides vanishes on the rim as the square root of the distance to the rim, in conformity with general theory.] Numerical results are shown in a number of curves for $ka=3, 4, 5, \sqrt{42}$. The calculated radiation resistance is in fair agreement with recent experiments [cf. Meier and Summers, Proc. Inst. Radio Engrs. 37, 609–616 (1949)]. The reviewer would remark that the radiation resistance of the isolated dipole antenna, that is when $ka=0$, is equal to $30\{\int_0^1 t^{-1}(1-\cos\pi t)dt-1\}$ $=19.45$ ohms, which makes extrapolation of the author's curve of Fig. 7 down to $ka=0$ very easy.

C. J. Bouwkamp (Eindhoven).

Gaponov-Grehov, A., and Miller, M. The excitation of a circular wave-guide by a ring antenna. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 19, 1260–1270 (1949). (Russian)

Formulas are given for the fields inside a circular waveguide excited by current flowing in a loop. The loop lies in a plane normal to the axis of the guide and has its center on the axis. *E. N. Gilbert* (Murray Hill, N. J.).

Ledinegg, E., und Urban, P. Über die Darstellung des vollständigen Systems der Eigenschwingungen beliebiger zylindrischer Hohlraumresonatoren mit horizontaler Schichtung des dielektrischen Mediums. Acta Physica Austriaca 3, 320–341 (1950).

It is known that the electromagnetic eigenvibrations of a simply connected cylindrical cavity belong either to electric ($H_z\equiv 0$) or magnetic type ($E_z\equiv 0$), where the axis is in the z-direction. A third ("Lecher") type with $H_z\equiv 0$, $E_z\equiv 0$ exists for doubly connected cavities. These results are extended to the case of dielectrically stratified cavities, $\epsilon/\epsilon_0=f(z)$, of importance for dielectric measurements at microwave frequencies. The proof is given for "smooth" $f(z)$ and then extended to the case of isolated discontinuities.

H. G. Baerwald (Cleveland, Ohio).

Zickner, G. Zur Berechnung der Randkorrektion von Kondensatoren. Arch. Elektrotechnik 38, 1–16 (1944).

This is a collection of expressions for the charge values given by various authors for parallel or orthogonal infinite and semi-infinite planes. Because these geometries can have practical significance for parallel plate condensers, the computation of capacitance values is added, stressing particularly the fringing effects. *E. Weber* (Brooklyn, N. Y.).

Ashour, A. A. The induction of electric currents in a uniform circular disk. Quart. J. Mech. Appl. Math. 3, 119–128 (1950).

By regarding a conducting surface of revolution as being made up of a sequence of coaxial annular cylinders, the author reduces the problem of the determination of the electric currents induced by an external field to the solution of a Fredholm integral equation. Methods of solution are discussed. *A. E. Heins* (Pittsburgh, Pa.).

Chambers, R. G. The conductivity of thin wires in a magnetic field. Proc. Roy. Soc. London. Ser. A. 202, 378–394 (1950).

The apparent resistivity of a conductor is increased when its linear dimensions become small because collisions with the surface reduce the mean free path. In a longitudinal magnetic field, however, the resistivity of a thin conductor decreases again because the spiral orbit lengthens the mean free path. A quantitative calculation of these effects is made by taking into account in the free-electron kinetic theory expression for the current the finite distance from any point in the conductor to the wall, both in the absence and in the presence of a longitudinal magnetic field. The possibility that some electrons may be specularly reflected from the surface (i.e., without loss of the drift velocity acquired in the electric field) is considered briefly. The conductivity of a thin wire, which contains an integration over the cross section of the conductor, is evaluated by numerical methods except in the presence of a large magnetic field or small values of orbit radius/conductor radius. In this latter case the theory predicts effective conductivities very close to the bulk value even for wires whose radius is very small compared to the bulk mean free path. A comparison with the results of experiments on sodium wires gives satisfactory agreement in view of the fact that the theory is based on a free-electron model and neglects the bulk magneto-resistance effect. *R. Karplus* (Cambridge, Mass.).

Redheffer, R. M. Remarks on the basis of network theory. J. Math. Physics 28, 237–258 (1950).

The paper contains a theory of linear "objects" connected in tandem. An object is a sort of generalized four-pole network specified only by four numbers t, τ, r, ρ representing transmission and reflection coefficients to left and right. The reciprocity law $t=\tau$ is not assumed; any set of four numbers is an object. A multiplication law for objects is so defined that the product of two objects has transmission and reflection coefficients given by the formulas one would normally use for two networks connected in tandem. Various sets of objects are shown to be groups. The main problem is to characterize as concisely as possible the objects which can reasonably be called shunt, lossless, or passive objects. For example, a set of lossless objects is defined to be a complete set of objects such that (i) $|t|^2+|r|^2=|\tau|^2+|\rho|^2=1$, (ii) the product of any two objects in the set is in the set, (iii) if $(t r \tau \rho)$ is in the set, so is $(\tau t \rho r)$. It then follows that only one such set exists and that a weak reciprocity law $|t|=|\tau|$ holds for lossless objects. *E. N. Gilbert.*

Duffin, R. J. Nonlinear networks. IV. Proc. Amer. Math. Soc. **1**, 233–240 (1950).

The electrical networks considered have nonlinearity because of the presence of transformers with ferro-magnetic cores (but free of hysteresis). The networks contain, besides the transformers, Ohmic resistors. The uniqueness properties of these networks when supplied with electromotive forces are deduced from earlier results of the author [Bull. Amer. Math. Soc. **52**, 833–838 (1946); **55**, 119–129 (1949); these Rev. **8**, 244; **10**, 536]. *N. Levinson.*

Cohn, R. M. The resistance of an electrical network. Proc. Amer. Math. Soc. **1**, 316–324 (1950).

The resistance of a two-terminal resistance network (or a general passive network without mutual impedances) is shown to have the following two properties, which are also shown to be equivalent. (i) Its partial derivative with respect to any branch resistance is equal to the square of the current in that branch for unit current through the network; (ii) its limit as the branch resistances approach any preassigned set of values is independent of the order in which the approach is made. The hope that one of these properties along with obvious requirements might serve to completely characterize the network is shown false by a counter-example due to H. W. Becker. *J. Riordan* (New York, N. Y.).

Gorn, Saul. Mathematical tools in the theory of four terminal nets. Army Air Forces, Air Technical Service Command, Engineering Division. Mem. Rep. no. **TSELP 5-29.** 81 pp.; appendices, 46 pp. (1946).

This is a thoroughgoing summary of transducer (two terminal-pair network) theory (the terminology in the title, four terminal net, is usual but confusing since the terminals are paired at sending and receiving ends, and a four terminal net exists apart from pairing). Topics considered are as follows: the general character of the bilinear transformation (which represents the transducer); conditions for physical realizability; transmission line circle diagrams; physical invariants (iterative and image impedances and transfer constants); power transfer, filter, matching, and phasing; the nature of the extension to n terminal-pairs. The statements are clear and careful and comprise a pleasant introduction for mathematical readers. *J. Riordan.*

Ming, Nai-Ta. Verwirklichung von linearen Zweipolschaltungen vorgeschriebener Frequenzabhängigkeit unter Berücksichtigung der Verluste von Spulen und Kondensatoren. Arch. Elektrotechnik **39**, 359–387 (1949).

The problem of realizing a prescribed function as the driving point impedance of a network constituted with linear ideal inductances, capacitances, and resistances was first solved by Brune [J. Math. Physics **10**, 191–236 (1931)]. Brune showed that necessary and sufficient conditions that a function $Z(\lambda)$ of the complex frequency variable λ shall be the driving point impedance of a finite network of such lumped elements are: (1) $Z(\lambda)$ is a rational fraction; (2) $Z(\lambda)$ is real when λ is real; (3) $\Re[Z(\lambda)]\geqq 0$ when $\Re[\lambda]\geqq 0$. A function satisfying (2) and (3) is commonly termed a "positive real function," although Brune included (1), at least by implication, when he coined the definition. The Brune procedure for synthesis leads to a ladder network with the minimum number of elements.

Since ideal elements, that is inductances and capacitances without dissipational losses, are unattainable, Ming attacks the problem of synthesizing a driving point impedance with lossy elements. Each inductance must have in series a resistance, and each capacitance a resistance in parallel. This is substantially his definition of a lossy two-pole or "Verlustzweipol" (V.Z.). The corresponding impedance function $W(\mu)$ is termed a "Verlustfunktion" (V.F.). By definition, a V.S. $W(\mu)$ is the function obtained from a positive real function $Z(\lambda)$ through the transformation $\lambda=\mu+\epsilon$. Hence $W(\mu)=Z(\mu+\epsilon)$. It is then shown that necessary and sufficient conditions that $W(\mu)$ shall be a V.F. are: (a) $W(\mu)$ is rational, and real when μ is real; (b) W is regular in the closed right half μ-plane, with the exception that a simple pole may occur at infinity; (c) $\Re[W(\mu)]>0$ when $\Re[\mu]=0$ except that at infinity a simple zero is allowed. In this case the limiting value of $\Re[W^{-1}]$ as μ approaches infinity must be positive. The proof follows mainly from known properties of positive real functions.

The rest of part I is concerned with the proof that the driving point impedance function of every V.Z. is a V.F., and conversely, to every V.F. there corresponds a lossy network V.Z. As is usually the case in network theory, the sufficiency proof, that is, the proof that to every permissible function there corresponds a network, is a method for constructing the network. Here the networks obtained are in general not ladder networks, but are obtained by series and parallel combinations of two-poles and four-poles. Inductive coupling is sometimes employed. Part II is concerned with several numerical illustrations, leading in some cases to lattice networks or combinations of lattice networks.

R. Kahal (St. Louis, Mo.).

Ming, Nai-Ta. Existenzbeweis zur Realisierung einer Verlustfunktion in linearen Wechselstromschaltungen durch Kettenschaltung von Verlustschaltelementen. Sci. Rep. Nat. Tsing Hua Univ. Ser. A. **5**, 350–377 (1949).

In the paper reviewed above the author established necessary and sufficient conditions for a V.F. $W(\mu)$ to be the driving point impedance of a lossy two-pole network V.Z. In this paper the author has succeeded in demonstrating that a V.F. $W(\mu)$ can always be realized by a ladder network of lossy elements. The method of synthesis is generally a modification of the Brune procedure. A sequence of five theorems serves as a set of directions to effect the synthesis in all possible cases. Theorem 1 corresponds closely to the manner in which a pole or zero occurring at infinity would normally be removed and would appear as a network element. However, instead of taking out merely a series inductance or shunt capacitance, these elements now appear with their associated series resistance or parallel conductance. Theorem 3 deals with the behaviour of the function at the origin. The analogue of the Brune circuit with a zero at the origin is the lossy circuit with impedance function whose real part $u(x)$ has an absolute minimum only at the origin and satisfies the condition $u'(x)=0$ at $x=0$, where $x=\omega^2$ ($\mu=\sigma+i\omega$). Theorem 4 is essentially an inversion of theorem 3, dealing with the procedure to be followed when $W(\mu)$ is regular at infinity, but with its real part assuming an absolute minimum R_0 only at infinity with the condition $\lim_{\mu\to\infty}\Re[1/(W(\mu)-R_0)]>0$. Strictly speaking, this has no analogue in the Brune method. Theorem 2 effects a synthesis by employing coupled coils. If $u(x)$ assumes an absolute minimum at a finite point $x_0>0$, and if $d^2u(x)/dx^2|_{x=x_0}\neq 0$, then the familiar coupled coil circuit with, however, the associated resistances may be realized. The use of theorems 1–4 in each case results in a lossy three-pole terminated by a V.F. of lower degree and thus

leads eventually to the realization of the complete network. However, in the event that the given function fails to satisfy the conditions of the first four theorems, then the procedure of theorem 5 is to be followed. This leads to a network consisting of a resistance in parallel with a V.F., and so fails to reduce the degree of the given V.F. But it is shown that when the function is so modified, the synthesis may be continued by the procedure of theorem 2. Thus the entire procedure evolves upon the proper combination of several canonical three-terminal ladder networks. *R. Kahal.*

Ming, Nai-Ta. Verwirklichung von linearen Vierpolschaltungen vorgeschriebener Frequenzabhängigkeit unter Berücksichtigung übereinstimmender Verluste aller Spulen und Kondensatoren. Arch. Elektrotechnik **39**, 452–471 (1949).

The problem treated is the realization of "reactive" two terminal pair networks, terminated in specified resistances, where it is supposed that the elements of the networks, inductance and capacitance, are not ideal but rather have certain ohmic losses. Losses in the elements are accounted for by assuming a resistance R in series with each inductance L, and a conductance G in parallel with each capacitance C. Certain results are obtained under a rather restricted set of conditions. Also, R, L, G, and C are supposed to be related in such a way that $R/L=G/C=\epsilon$ is constant for all such elements. It is assumed further that the corresponding network with lossless elements can be realized without inductive coupling. With these conditions it is shown how the desired network with lossy elements may sometimes be realized completely. In other cases an approximation can be made. There is also a discussion of the conditions under which a low-pass filter with lossless, reactive elements may be realized without inductive coupling. Numerical examples and a procedure for calculations are included.

R. Kahal (St. Louis, Mo.).

Mather, Norman W. An analysis of triple-tuned coupled circuits. Proc. I.R.E. **38**, 813–822 (1950).

Three parallel-resonant circuits in cascade arrangement are considered with the following restrictions: magnetic coupling exists only between circuits 1 and 2 and between 2 and 3; the coupling coefficients are small; all three circuits have individually the same angular resonance frequency $\omega_r=2\pi f_r$; the dissipation factors are small but different in the three circuits; only symmetrical solutions with respect to the resonance frequency f_r are considered; a frequency deviation variable $x=4\Omega/D_T$ is used where $\Omega=(\omega/\omega_r-\omega_r/\omega)$ and is assumed very small, and where D_T is the sum of the three circuit dissipation factors. The transfer impedance $Z_T=V_3/I_1$ (V_3 output voltage, I_1 input current) is then in first order approximation in the neighborhood of the resonance frequency inversely proportional to the quantity $T=[2x^3-(f^2+1)^2+4g^4]+j[x^3-2(f^2+1)x]$, with f and g as parameters independent of frequency (only in first order approximation). The main part of the paper discusses in considerable detail the function T^{-1} whose magnitude can exhibit three equal peaks which fact is of greatest interest in connection with broad-band circuit response. Transition from the one-peak to the three-peak characteristic occurs for $f^2=0$ and is interpreted in terms of circuit requirements. The symmetrical asynchronous case, where circuits 1 and 3 have resonance frequencies symmetrically above and below that of circuit 2, is also discussed. *E. Weber.*

Rozenfel'd, B. A. The symbolic method and vector diagrams for nonsinusoidal currents. Trudy Sem. Vektor. Tenzor. Analizu **7**, 381–387 (1949). (Russian)

Let a linear network be driven by a nonsinusoidal voltage $v(t)$. The current flowing into the circuit is found with the aid of a vector diagram which is actually just a set of ordinary vector diagrams (the kind used in elementary alternating current theory), one for each Fourier component of $v(t)$. *E. N. Gilbert* (Murray Hill, N. J.).

Quantum Mechanics

Motchane, Léon. Sur l'irréversibilité du temps et la représentation des notions fondamentales de la mécanique. C. R. Acad. Sci. Paris **229**, 581–583 (1949).

The author contends that, because time is irreversible, time derivatives should be left-lateral (and not bilateral). It is observed that primitives I with left-derivative 0 have variation $I(b)-I(a)$ represented by a discrete set of constants, and it is concluded that quantification of physical quantities is a consequence of the use of left-derivatives and the irreversibility of time. A method is indicated for calculating quantum states from the saltus of a function having left-derivative 0. *C. C. Torrance* (Annapolis, Md.).

Motchane, Léon. Exemples d'applications de la représentation des notions fondamentales de la mécanique par des fonctions unilatérales. C. R. Acad. Sci. Paris **230**, 2264–2266 (1950).

The author illustrates his theory [see the preceding review] by setting up, without reference to the wave equation, a matrix for an harmonic oscillator, and explains the properties of an oscillator in terms of the singularities of functions having left-derivative 0. As a second example, the author discusses the motion of a mass-point on a sphere.

C. C. Torrance (Annapolis, Md.).

Soonawala, M. F. Circular and elliptical quantum orbits. Indian J. Phys. **24**, 95–102 (1950).

The theory developed by the author [same J. **21**, 137–142 (1947)] yields an inverse cube law of gravitational force between two particles, and the classical quantum theory for the motion of two particles under such a law of force is developed here. *H. C. Corben* (Pittsburgh, Pa.).

Maravall, Dario. Concerning the continuity equation of nonrelativistic quantum mechanics. Euclides, Madrid **10**, 112–115 (1950). (Spanish)

Sato, Iwao. An attempt to unite the quantum theory of wave field with the theory of general relativity. Sci. Rep. Tôhoku Univ., Ser. 1. **33**, 30–37 (1949).

The author writes down a covariant set of commutation relations and field equations for a real quantized scalar field U in a Riemannian space-time with metric tensor $g_{\mu\nu}$. The $g_{\mu\nu}$ are considered to be c-numbers and the field equation for U is linear; this part of the formalism is certainly consistent and has a clear physical interpretation. Secondly, the author writes down the field equations for the $g_{\mu\nu}$ in the Einsteinian form $G_{\mu\nu}-\frac{1}{2}g_{\mu\nu}G=-8\pi kT_{\mu\nu}(\Psi)$, where $T_{\mu\nu}(\Psi)$ is the expectation value of the energy tensor of the U field calculated in the quantum state Ψ. Thus the solutions $g_{\mu\nu}$ of the field equations depend in a nonlinear way on Ψ, and

the principle of superposition for quantum states is not valid. It is not clear whether the coupled equations for the quantized U and the classical $g_{\mu\nu}$ can be interpreted physically, or even whether they constitute a consistent formalism. The author does not claim to have overcome these difficulties. *F. J. Dyson* (Princeton, N. J.).

Nambu, Y. Derivation of the interaction potential from field theory. Progress Theoret. Physics **5**, 321–323 (1950).

The author discusses the difficulties involved in deriving from a relativistic quantized field theory the correct interaction between two sources of the field. These difficulties are distinct from the problem of divergences in the field theory. We are able to define in an unambiguous way, by means of an S-matrix, the interaction involved in a scattering of one free source by another. But there exists no unambiguous method of finding the interaction of sources in bound states, including effects of retardation and effects of higher order in the field coupling. The author proposes to define the interaction in all cases by the requirement that it should give rise to the correct S-matrix for scattering processes. Applying his definition to the case of electromagnetic interactions, he calculates the interaction between two particles up to terms of order e^4, neglecting terms representing short-range effects (interactions effective only within a range of the order of the particle Compton wavelength). In the static approximation this result reduces to the Breit interaction plus an additional term of order e^4. More details are to be published later. *F. J. Dyson.*

Umezawa, Hiroomi, and Kawabe, Rokuo. On the relativistically improved integration in perturbation theory. Its meaning in Feynman theory. Progress Theoret. Physics **5**, 266–271 (1950).

In a previous paper [same journal **4**, 420–422 (1949); these Rev. **11**, 762] the authors showed how to recast noncovariant perturbation theory into a form in which it yields formally covariant results. They here apply the same method to the covariant perturbation theory of Feynman, showing how their method of calculation is related to his. In all cases their method gives results identical with those of Feynman. *F. J. Dyson* (Princeton, N. J.).

Yukawa, J., Oda, N., and Umezawa, H. Relativistic covariance in the quantum electrodynamics. Progress Theoret. Physics **5**, 320–321 (1950).

The self-stress of the electron [see A. Pais and S. T. Epstein, Rev. Modern Physics **21**, 445–446 (1949)] ought to be zero in a correct relativistic theory. In the usual quantum electrodynamics the self-stress appears not to vanish when it is calculated by perturbation theory; its value is in fact not well defined, because of the divergence of the self-energy. The authors here verify that, in the theory of a point electron interacting with the electromagnetic field and with a hypothetical scalar field arranged so as to make the whole self-energy finite by a compensation of infinities, the self-stress vanishes identically as it should. *F. J. Dyson* (Princeton, N. J.).

Villars, F. On the energy-momentum tensor of the electron. Physical Rev. (2) **79**, 122–128 (1950).

Direct calculation of radiative corrections to the self-stress of an electron [Pais and Epstein, Rev. Modern Physics **21**, 445–446 (1949)] seems to give a nonvanishing result in contradiction to the conservation laws which require the stress tensor to be divergenceless. Rohrlich [Physical Rev. (2) **77**, 357–360 (1950)] has treated the question. This paper deals with the somewhat more general problem of the first order corrections to the energy momentum tensor. It is found that it is possible to "regularize" the calculation so as to obtain a divergenceless answer. However, the regularization has to be "realistic." One must introduce an additional neutral vector meson with an anti-Hermitian coupling. Explicit formulas for the correction to the energy momentum tensor are given. *K. M. Case.*

Feldman, David. On realistic field theories and the polarization of the vacuum. Physical Rev. (2) **76**, 1369–1375 (1949).

Pauli and Villars [Rev. Modern Physics **21**, 434–444 (1949); these Rev. **11**, 301] obtain unambiguous results in their "formalistic theory" by using a regularization procedure which guarantees the cancelling out of divergent terms in the expressions for self-energies and vacuum polarization currents. The present author investigates whether analogous results might be achieved in a "realistic theory" in which interactions of different particles with one another are taken into account. First he considers a model which corresponds closely to Pauli's scheme: a family of spinor fields interacting with a family of neutral vector fields (including the photon field), with vector coupling linking each spinor and vector field. It turns out that one has to use imaginary coupling constants in order to obtain convergent expressions for the self-energies and vacuum polarization currents. This has the consequence that the Hamiltonian is not Hermitian and probability is not conserved. A simple physical interpretation is thus made impossible. An attempt to consider more general mixtures of particles also leads to failure. To show this the current density which is induced in the vacuum by an external electromagnetic field is calculated to order e^3 for charged matter fields of scalar, spinor, and vector type, with and without anomalous magnetic moments. Some of the divergent terms in the expressions obtained have identical sign, and others no counterpart, so that full compensation is not possible although the photon self-energy by itself may be made to vanish. From these results the author concludes that a realistic approach has no chance to succeed unless more general field theories are used instead of the usual linear field theories. *E. Gora* (Providence, R. I.).

Katayama, Yasuhisa, Sawada, Katuro, and Takagi, Shuji. Five dimensional approach to regularized quantum electrodynamics. Progress Theoret. Physics **5**, 14–24 (1950).

Feldman [see the preceding review] has shown that Pauli's regularization procedure [Pauli and Villars, Rev. Modern Physics **21**, 434–444 (1949); these Rev. **11**, 301] corresponds to a theory of spinor fields interacting with neutral vector fields of nonvanishing rest mass, the so called "mixed theory." The authors propose a five-dimensional generalization of the quantized field theory for electrons and light quanta in interaction and obtain a Hamiltonian containing expressions which are interpreted as density distributions of higher mass states. These higher mass states are the essential feature of the theory in the same way as the neutral vector particles of nonvanishing rest mass in the "mixed theory." The five-dimensional formalism is used to calculate self-energies of the electron and the photon, the vacuum polarization, and the anomalous magnetic moment of the electron in second order approximation. The results

obtained agree with Pauli's. The equivalence of the five-dimensional formalism and Pauli's procedure is thus demonstrated in this approximation. In higher approximations, on the contrary, differences are to be expected. The difficulties encountered there are left for future investigations.
E. Gora (Providence, R. I.).

Katayama, Yasuhisa. Problems of ambiguity in quantum field theory. Progress Theoret. Physics 5, 272–282 (1950).

Evaluation of matrix elements for complex processes in quantum field theory typically gives rise to ambiguous results. Well-known examples are the photon self-energy, and the amplitudes for decay of various kinds of neutral mesons into photons [Fukuda et al., same journal 4, 477–484 (1949)]. In the present paper the various types of ambiguous integrals arising in such processes are listed systematically. It is shown that in all cases the ambiguities belong to a small number of special forms. The two methods at present available for resolving the ambiguities are discussed. These are: (1) the regulator method of Pauli and Villars [Rev. Modern Physics 21, 433–444 (1949); these Rev. 11, 301]; and (2) the use of formal requirements, which demand that the results of calculations be gauge-invariant and satisfy various relations of equivalence. Method (2) is not adequate to deal with all ambiguities, and method (1) is itself ambiguous since it may be applied in different ways to give different results. The authors conclude that no generally satisfactory method of eliminating the ambiguities from field-theoretical calculations exists at present.
F. J. Dyson (Princeton, N. J.).

Hori, S., and Sawada, K. On the anomalous magnetic moment of nucleon in vector and pseudovector meson theory. Progress Theoret. Physics 5, 333–334 (1950).

The different results obtained by Luttinger [Physical Rev. (2) 75, 1277–1278 (1949)] and Sawada [same journal 4, 383–385 (1949)] for the anomalous nuclear magnetic moment in the vector and pseudovector meson theories is attributed to the use of different expressions for the meson current. Sawada's use of Stückelberg's formalism implies a meson magnetic moment that differs from the usual one of a charged vector field; it was obtained by adding a Pauli-term to the Lagrangian.
R. Karplus.

Nishijima, Kazuhiko. On the integrability condition in the "super-many-time theory." Progress Theoret. Physics 5, 187–195 (1950).

If a quantity is conserved in the one-time theory, it does not follow that it is conserved in the many-time theory, and in particular the energy-momentum conservation law according to the latter requires the integrability condition for its proof. For points a finite distance apart on a space-like surface the condition reduces to the commuting of the Hamiltonians at these two points and from this condition it is argued that if in process i there are $n_{i\alpha}$ fermions of type α involved, and if their wave functions commute for different α, then two processes i, j are incompatible unless $\sum_\alpha n_{i\alpha} n_{j\alpha}$ is even. According to this the processes $\pi \to \mu + \nu$, $\mu \to e + 2\nu$ are incompatible.
H. C. Corben.

Kwal, Bernard. Contribution à la théorie des champs non localisables. J. Phys. Radium (8) 11, 213–218 (1950).

The author discusses the derivation of a set of spinor equations for a nonlocalizable spinor wave function which differ from those proposed by Yukawa [Physical Rev. (2) 76, 300–301, 1731 (1949)] in two respects: (1) They involve eight component spinors, and (2) the solutions have the property that they vanish outside a sphere of radius λ ($r_\mu r^\mu = \lambda^2$) in the $X_\mu = \frac{1}{2}(x_\mu + x_\mu')$, $r_\mu = \frac{1}{2}(x_\mu - x_\mu')$ representation.
A. H. Taub (Urbana, Ill.).

Destouches-Février, Paulette. Sur la recherche de l'équation fonctionnelle d'évolution d'un système en théorie générale des prévisions. C. R. Acad. Sci. Paris 230, 1742–1744 (1950).

The author gives a justification for the hypothesis that the operator $\mathfrak{U}$ of J.-L. Destouches is governed by a hereditary law in the case of a system having a variable number of components.
C. C. Torrance (Annapolis, Md.).

Visconti, Antoine. Équation intégrale opératorielle d'évolution d'un système physique. C. R. Acad. Sci. Paris 230, 1744–1746 (1950).

The author develops, on the basis of a hereditary law, an integral equation for the operator $\mathfrak{U}$. This equation provides the matrix S of Heisenberg and leads to certain relations of Feynman.
C. C. Torrance (Annapolis, Md.).

Destouches, Jean-Louis. Quelques propriétés de l'équation intégrale opératorielle d'évolution. C. R. Acad. Sci. Paris 230, 1747–1749 (1950).

From Visconti's integral equation for $\mathfrak{U}$ [see the preceding review], the author obtains an integral equation for the energy W of a conservative system, and introduces the concept of hereditary action.
C. C. Torrance.

Nogami, Mokichiro. On the lattice polarization induced by electronic motion. Progress Theoret. Physics 5, 56–64 (1950).

In der Schrödingergleichung des Problems

$$\left(-\frac{h^2}{8\pi^2 m}\Delta + H_e + H_l + V + V'\right)\Psi = -\frac{h}{2\pi i}\frac{\partial \Psi}{\partial \tau}$$

bedeutet H_e die Energie der Kristallelektronen, H_l die Schwingungsenergie des Gitters und die Störterme V und V' die elektrostatische Wechselwirkung des einfallenden Elektrons mit den Kristallelektronen, Ionen, mit dem von den Ionenschwingungen herrührenden Polarisationsfelde und die Wechselwirkung der Kristallelektronen mit dem Polarisationsfelde. Zur Lösung wird Ψ in das Produkt der Eigenfunktionen des einfallenden Elektrons, der Kristallelektronen und der Gitterschwingungen zerlegt. Zur weiteren Berechnung werden die harmonischen Gitterschwingungen durch ihre Normalkomponenten ausgedrückt und die Eigenfunktionen der Gitterelektronen in der Blochschen Näherung benutzt. Gegenüber dem bekannten und halbklassisch hergeleiteten Resultate von Landau bzw. Gurney und Mott [Proc. Phys. Soc. 49, 32–35 (1937)] in dem ein Faktor $(1/\kappa_0 - 1/\kappa)$ auftritt, wird dieser Faktor nach den Rechnungen des Verfassers durch den Ausdruck $(\kappa - 1)$ ersetzt, wo κ die ganze Dielektrizitätskonstante und κ_0 dieselbe unter Weglassung der von den Ionenverschiebungen herrührenden Gliedes bedeuten. Im letzten Paragraphen wird gezeigt, dass man bei der Berechnung von Anregungen im Gitter nicht immer mit dem Begriffe der klassischen Dielektrizitätskonstante auskommen kann.
T. Neugebauer.

Nogami, Mokichiro. On an approximate solution of the many body problem of Fermi-Dirac particles. Progress Theoret. Physics 5, 65–71 (1950).

Starting with a second quantized Hamiltonian and an unperturbed state corresponding to a completely degenerate

Fermi gas an attempt is made to solve the nuclear many body problem by expanding in inverse powers of the number of nucleons. All terms in the Hamiltonian except those diagonal or corresponding to collision of two particles below the Fermi surface (and the inverse process) are dropped as small. By introducing as variables the number of holes below and particles above the surface and dropping quadratic terms, a rigorously soluble equation for the wave function is obtained. Calculating the binding energy of a heavy nucleus with Wigner, Majorana, Heisenberg and Bartlett forces gives a result similar in form to that found by second order perturbation theory but somewhat smaller in absolute magnitude. *K. M. Case* (Rochester, N. Y.).

Moshinsky, Marcos. Interference phenomena for particles obeying Bose or Fermi statistics. Proc. Amer. Philos. Soc. **94**, 53–58 (1950).

According to both classical and quantum optics, light beams emitted from two different sources do not interfere at the same point. However, when they consist of short trains of nonmonochromatic waves, they will, according to classical optics, show secondary interference when observed at two different points: there will be a phase-independent correlation between the intensities at the two points. The paper studies this phenomenon on a quantum mechanical basis. It is present with both Bose-Einstein and with Fermi-Dirac statistics, but agrees in form with the classical situation only in the Bose-Einstein case. *B. O. Koopman.*

Pluvinage, Ph. Fonction d'onde approchée à un paramètre pour l'état fondamental des atomes à deux électrons. Ann. Physique (12) **5**, 145–152 (1950).

The author applies the Ritz method to find approximately the lowest energy level (from the Schrödinger equation) for a helium-like atom with two electrons. He uses a new approximate wave function $u_k(r_{12})\exp(-Zr_1-Zr_2)$ depending on a single parameter k. Here r_{12} is the distance between electrons, r_1 and r_2 the distances from electrons to nucleus, and the function $u_k(r)$ is a solution of the differential equation $ru_k''+2u_k'+(rk^2-1)u_k=0$. In the resulting calculation $u_k(r_{12})$ is expressed in terms of the hypergeometric function. The calculated results for eight atoms from helium to ionized fluorine are compared with experimental values. They are found to be more accurate than those obtained by other authors, who used approximate wave functions depending on two parameters. *O. Frink.*

Maslov, P. G. On the determination of the inverse matrices of the potential energy of polyatomic molecules. (Refined method of steepest descent). Doklady Akad. Nauk SSSR (N.S.) **71**, 867–870 (1950). (Russian)

The purpose of the author is to find the energy levels of the vibration spectrum of a molecule. This involves the general question of solving the eigenvalue problem $(\lambda G-A)x=0$, where x is a vector and G and A are operators in a finite linear space with Hermitian metric. The method adopted is a refinement of the method of steepest descent [cf. Temple, Proc. Roy. Soc. London. Ser. A. **169**, 476–500 (1939)] which the author asserts is extremely convenient for calculation and in one step achieves the accuracy of the third or fourth step of the unrefined method.

A. J. Coleman (Toronto, Ont.).

Fradkin, E. S. On the theory of particles with large spin. Akad. Nauk SSSR. Žurnal Eksper. Teoret Fiz. **20**, 27–38 (1950). (Russian)

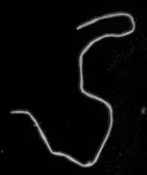
The author's summary is as follows. "This paper studies the problem of obtaining explicitly spinor-tensor equations for particles of arbitrary given spin and discusses their uniqueness and their splitting in the absence of a field. The paper seeks to make available to the practical needs of the physicist the related work of Gel'fand and Yaglom [Doklady Akad. Nauk SSSR (N.S.) **59**, 655–658 (1948); these Rev. **9**, 496]. The cases of particles with spin 5/2 and 3/2 are discussed in detail." *A. J. Coleman* (Toronto, Ont.).

Thermodynamics, Statistical Mechanics

Prigogine, I. Le domaine de validité de la thermodynamique des phénomènes irréversibles. Physica **15**, 272–284 (1949).

There are many investigations on the application of thermodynamic methods to irreversible processes [cf., e.g., L. Onsager, Physical Rev. (2) **38**, 2265–2279 (1931)]. For this method it is essential that the Gibbs fundamental relation connecting the differential of the entropy with that of the energy, volume, and other extensive variables should be applicable even in the case of irreversible processes. Starting from the Chapman-Enskog method [e.g., S. Chapman and T. G. Cowling, The Mathematical Theory of Non-uniform Gases, Cambridge University Press, 1939; these Rev. **1**, 187], the author shows that this assumption is justified in the so-called second approximation of this method, but not in general. The validity of the method extends also to chemical reactions which are slow enough not to disturb considerably the equilibrium distribution of each constituent. *L. Tisza* (Cambridge, Mass.).

Gürsey, Feza. Classical statistical mechanics of a rectilinear assembly. Proc. Cambridge Philos. Soc. **46**, 182–194 (1950).

The classical thermodynamic properties are computed for a one-dimensional gas of spherical molecules with short range attractive forces. The method is based on the fact that when interaction between neighbors only is taken into account, the phase integral assumes an iterated form and can be evaluated by means of the Laplace transformation. For noninteracting elastic spheres the results reduce to those of Tonks [Physical Rev. (2) **50**, 955–963 (1936)]. In agreement with the general results of Rushbrooke and Ursell [Proc. Cambridge Philos. Soc. **44**, 263–271 (1948); these Rev. **10**, 276] the pressure is an analytic function of the temperature and length, and hence no critical points result. Nevertheless the model behaves like a perfect gas at high temperatures and like a crystal near absolute zero. At very low temperatures the slope of the isothermal curves can suffer extremely sharp changes which may be interpreted as changes of phase. *L. Tisza* (Cambridge, Mass.).

Wataghin, G. Irreversible processes and the formation of nuclei. Nuovo Cimento (9) **6**, Supplemento, no. 2 (Convegno Internazionale di Meccanica Statistica), 241–243 (1949).